Lecture notes on Social Security

Massimiliano Menzietti
A.Y. 2019/2020
Week 3
Individual present values
Individual present value

In order to calculate the present values of the contributions payed by active insured to the pension scheme and the present values of the pension benefits that are payed by the pension schemes to beneficiary, we need to define the present values for a single life.
Present value of salaries

It is necessary in order to compute the present value of contributions:

\[
PVS(x, t) = \sum_{y=x}^{r-1} \frac{s(y, t + y - x)}{s(x, t)} \cdot y-xp_x^{aa} \cdot v^{y-x}
\]

for \( b \leq x < r \), where

\[
y-xp_x^{aa} = \frac{L_y^a}{L_x^a}
\]

\[
v = \frac{1}{1 + i}
\]

with

\( s(x, t) \) is a function that represents the average salary of the whole active population aged \( x \) at time \( t \)

\( \{L_x^a\} \) is the survival table for active insured

\( i \) is the interest rate

Both the two last technical bases are assumed to be time invariant.
Present values of retirement pensions for a retiree

Present values of retirement pensions for a retiree

\[
PVR^p(x) = \sum_{y=x}^{\omega-1} y-x p_x^{pp} \left( \frac{1 + \beta}{1 + i} \right)^{y-x}
\]

for \( x \geq r \), where

\[
y-x p_x^{pp} = \frac{L_y^p}{L_x^p}
\]

with

\( \{L_x^p\} \) is the survival table for a retiree

\( \beta \) is the pension indexation rate

Both the technical bases are assumed to be time invariant.
Present values of retirement pensions for a retiree

Setting

\[
\frac{1}{1 + j} = \frac{1 + \beta}{1 + i} \rightarrow j = \frac{1 + i}{1 + \beta} - 1 = \frac{i - \beta}{1 + \beta}
\]

and

\[
v_j = \frac{1}{1 + j}
\]

We have

\[
PVR^p(x) = \sum_{y = x}^{\omega - 1} y-x p_x^{pp} v_j^{y-x}
\]

observe that for small values of \(\beta\) is \(j \approx i - \beta\)
Present values of retirement pensions for an active insured

hypothesis 1) old-age retirement only at exact age $r$

\[ PVR^a(x, t) = P(r, s, t + r - x) \frac{s(r, t + r - x)}{s(x, t)} r_x p_x^{aa} v^{r-x} PVR^p(r) \]

for $b \leq x < r$, where

$P(r, s, t)$ is a function that represents the retirement pension as a proportion of the final salary, it depends on retirement age $r$, on the computation year $t$ and on the vector of salary $s$.

hypothesis 2) old-age retirement only at age $y$ (with $r^* \leq y \leq r$

\[ PVR^a(x, t) = \sum_{y=r^*}^{r} P(y, s, t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y_{-1-x} p_x^{aa} q_{y-1}^{ap} v^{y-x} PVR^p(y) \]
Present values of invalidity pensions

Present values of invalidity pensions for an invalid

\[ PVI^i(x) = \sum_{y=x}^{\omega-1} y-x p_x^i v_j^{y-x} \]

for \( x \geq b \), where

\[ y-x p_x^i = \frac{L^i_y}{L^i_x} \]

with

\( \{L^i_x\} \) is the survival table for a invalidity pensioners, assumed to be time invariant.

Present values of invalidity pensions for an active insured

\[ PVI^a(x, t) = \sum_{y=x}^{r-1} Pi(y, s, t + y - x) \frac{s(r, t + r - x)}{s(x, t)} y-x p_x^{aa} i_y v^{y+1-x} PVI^i(y + 1) \]

for \( b \leq x < r \), where

\( Pi(y, s, t) \) is a function that represents the invalidity pension as a proportion of the final salary, it depends on retirement age \( r \), on the evaluation year \( t \) and on the vector of salary \( s \)

\( i_y \) is the probability to become disable at age \( y \)
Present values of survivor pensions for the widow/widower (orphans)

Present values of survivor pensions for the widow/widower

\[ PVF^w(x) = \sum_{y=x}^{\omega-1} y-x p^w_{x} v^y_{j}^{y-x} \]

for \( x \geq 0 \), where

\[ y-x p^w_{x} = \frac{L^w_y}{L^w_x} \]

with

\{L^w_x\} is the survival table for a widow/widower, assumed to be time invariant.

Present values of survivor pensions for the orphans

A similar expression can be obtained for an orphan

\[ PVF^o(x) = \sum_{y=x}^{k-1} y-x p^o_{x} v^y_{j}^{y-x} \]

with \( k \) maximum age at which the benefit is payed to an orphan.
Present values of widows/widowers pensions for the widow/widower (orphans)

If the family unit is composed by more than one person we use a last survivor annuity formula, for example in the case of a widow aged $x$ and an orphan aged $y$ we have:

$$PVF_{w,o}^{\omega-x-1}(x; y) = \theta_w \sum_{t=0}^{k-x-1} t p_{x}^{ww} (1 - t p_{y}^{oo}) v_j^t + \theta_o \sum_{t=0}^{k-y-1} t p_{y}^{oo} (1 - t p_{x}^{ww}) v_j^t +$$

$$+ \theta_{w;o} \sum_{t=0}^{k-y-1} t p_{x}^{ww} t p_{y}^{oo} v_j^t$$

where

- $\theta_w$ is the proportion of the actual potential pension payed to a spouse alone
- $\theta_o$ is the proportion of the actual potential pension payed to an orphan alone
- $\theta_{w;o}$ is the proportion of the actual potential pension payed to a survived family composed by a spouse and an orphan
Present values of widows/widowers pensions for the widow/widower (orphans)

Obviously we have

\[
PVF_{w;o}(x; y) = \theta_w \sum_{t=0}^{\omega-x-1} t p_{x}^{ww} v_j^t + \theta_o \sum_{t=0}^{k-y-1} t p_{y}^{oo} v_j^t +
\]

\[
+ (\theta_{w;o} - \theta_w - \theta_o) \sum_{t=0}^{k-y-1} t p_{x}^{ww} t p_{y}^{oo} v_j^t
\]
Present values of survivor pensions (death in service) for an active insured

Usually the present value of survivor pensions is determined assuming that only the widow/widower survive adopting an age correction in order to reduce the underestimate produced ignoring orphans:

\[
PVFa(x, t) = \sum_{y=x}^{r-1} P(y, s, t + y - x) \frac{s(r, t + r - x)}{s(x, t)} y-x p_x^{aa} q_y^{aw} v^{y+1-x} \theta_F PVF^w(y + 1 + c)
\]

for \( b \leq x < r \), where

\( q^{aw} \) depends on \( q^a_x \) (the active death probability) and \( w_y \) (the probability that the active insured leave a family)

\( \theta_F \) is proportion of the actual potential pension payed to a survived family obtained as a mean of the different proportion for different family composition.

\( c \) is a correction on the spouse age in order to consider the impact of the orphans pensions.
Present values of survivor pensions (death in retirement)

Present values of survivor pensions (death in retirement) for a retired person

\[ PVF^p(x) = \sum_{y=x}^{\omega-1} y-x p_x^{pp} q_y^{pw} v^{y+1-x} \theta_F PVF^w(y + 1 + c) \]

for \( r \leq x < \omega \), and where

\( q_x^{pw} \) depends on \( q_x^p \) (the retired death probability) and \( w_y \) (the probability that the retired leave a family)

A similar expression is obtained for invalidity pensioners.
Present values of survivor pensions (death in retirement)

Present values of survivor pensions (death in retirement) for an active insured: hypothesis 1) old-age retirement only at exact age $r$

$$PVF^{a,p}(x, t) = P(r, s, t + r - x) \frac{s(r, t + r - x)}{s(x, t)} r_{x}p_{x}^{aa} v^{r-x} PVF^{p}(r)$$

for $b \leq x < r$.

Present values of survivor pensions (death in retirement) for an active insured: hypothesis 2) old-age retirement only at age $y$ (with $r^* \leq y \leq r$)

$$PVR^{a}(x, t) = \sum_{y=r^*}^{r} P(y, s, t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y_{y-1}p_{x}^{aa} q_{y-1}^{ap} v^{y-x} PVF^{p}(y)$$
Salary and benefit functions
Salary function

Previously a function \( s(x, t) \) that represents the average salary of the whole active population aged \( x \) at time \( t \) has been defined.

The classical method for the projection of the insured salary, refers to age- and time-related average salaries which are projected, allowing for the progression of each cohort's average salary according to an age-related salary scale function (or merit salary scale) \( ss_x \) and taking into account the escalation of the general level of salaries, but the method assumes an invariant salary scale function.

Although the starting salaries of new entrant cohorts could be varied, this method does not permit the modelling of the variation over time in the age-wise salary structure of the active population. Nor does it allow the salary distribution at each age to be taken into account.
Salary function

\[ s(x, t) = s(x - 1, t - 1) \frac{SS_x}{SS_{x-1}} (1 + \gamma_I(t))(1 + \gamma_P(t)) \]

where

\( SS_x \) is the age-related salary scale function (merit salary scale);
\( \gamma_I(t) \) is the rate of inflation reflected in the salary increase;
\( \gamma_P(t) \) is the rate of productivity reflected in the salary increase.

Assuming \( b \) as minimum entry age and \( \gamma_I(t) \) and \( \gamma_P(t) \) constant over time the following expression holds:

\[ s(x, t) = s(b, t - x + b) \frac{SS_x}{SS_b} [(1 + \gamma_I)(1 + \gamma_P)]^{x-b} \]

where

\[ s(b, t) = ss_b [(1 + \gamma_I)(1 + \gamma_P)]^{t-t_0} \]

so we have

\[ s(x, t) = ss_x [(1 + \gamma_I)(1 + \gamma_P)]^{t-t_0} \]
Benefit functions

The benefit function is used to determine the amount of benefit paid at retirement, disablement or death.

Let $b_x$ denote the annual benefit accrual during age $x$ to age $x + 1$ for an age-$b$ entrant, we refer to as the benefit accrual function.

The accrued benefit, denoted by $B_x$, is equal to the sum of each attained age accrual up to, but not including, age $x$. This function is called the accrued benefit function and is defined by

$$B_x = \sum_{t=b}^{x-1} b_t$$

Obviously $B_r$ represents the benefit paid at retirement.

We can have different evolution of the benefit functions.
Benefit functions: Flat unit benefit

Under a flat unit benefit formula $b_x$ is equal to a flat annual benefit payable per year of service. Although the attained age subscript is shown here, it is unnecessary since the accrual is independent of age.

The accrued benefit is a years-of-service multiple of the benefit accrual

$$B_x = (x - b)b_x$$

So

$$B_r = (r - b)b_x$$

If the final benefit $B_r$ is predefined (given the assumed year of service $r - b$), $b_x$ is equal to

$$\frac{B_r}{(r - b)}$$
Benefit functions: Career average

The career average benefit formula has the following definitions for the benefit accrual and the accrued benefit functions at age \( x \):

\[
b_x = ks_x \\
B_x = kS_x
\]

where \( s_x \) is the salary at age \( x \) and \( S_x \) represents the cumulative salary from entry age \( b \) up to, but not including, age \( x \):

\[
S_x = \sum_{t=b}^{x-1} s_t
\]

The benefit functions under the career average formula follow precisely the pattern of the attained age salary and the cumulative salary.

The benefit at retirement will be

\[
B_r = kS_r
\]

if we fix the retirement benefit \( B_r \) and assume a cumulative salary evolution \( S_r \),

\[
k = \frac{B_r}{S_r}
\]
Benefit functions: Final average

The final average benefit formula is somewhat more complicated. Let $n$ denote the number of years over which the active’s salary prior to retirement is to be averaged, and let $k$ equal the proportion of the average salary provided for year of service. The projected retirement benefit, assuming retirement occurs at the beginning of age $r$, is defined as

$$B_r = k(r - b) \frac{1}{n} \sum_{t=r-n}^{r-1} s_t = k(r - b) \frac{1}{n} (S_r - S_{r-n})$$

The attained age benefit accrual and the accrued benefit functions can be defined in several ways under this benefit formula.

One approach is to define $B_x$ according to the benefit formula based on the participant’s current salary average

$$B_x = k(x - b) \frac{1}{n} (S_x - S_{x-n})$$

Where $n$ is the smaller between the years specified in the benefit formula or $x - b$. 
Benefit functions: Final average

The corresponding benefit accrual at age \(x\) can be determined by the following:

\[
b_x = B_{x+1} - B_x
\]

\[
= k \frac{1}{n} (S_{x+1} - S_{x+1-n}) + k(x - b) \frac{1}{n} [(S_{x+1} - S_{x+1-n}) - (S_x - S_{x-n})]
\]

\[
= k \frac{1}{n} (S_{x+1} - S_{x+1-n}) + k(x - b) \frac{1}{n} [s_x - s_{x-n}]
\]

The first term is the portion of the benefit earned in the current year based on the participant’s current \(n\)-year salary average, while the second term represents the portion earned(loosed) as a result of the increase(decrease) in the \(n\)-year salary average base.

The increase in salary base \((\frac{1}{n} [s_x - s_{x-n}])\) is multiplied by years-of-service to date. Therefore, the benefit accrual during age \(x\) includes an implicit updating of previous accruals which can cause \(b_x\) to be a steeply increasing function of \(x\).
Benefit functions: Final average

As we will see later, steeply increasing benefit accrual and accrued benefit functions produce even steeper pension cost function under some methods of determining pension costs and liabilities, characteristics that may be undesirable. Consequently, two modifications to benefit functions that are based on the plan’s formula have been developed in order to mitigate this effect: the constant unit modification and the constant percent modification.

The constant unit modification defines the benefit accrual function as a pro rata share of the participant’s retirement-age projected benefit:

\[ cu\, b_x = \frac{B_r}{r - b} = k \frac{1}{n} (S_r - S_{r-n}) \]

That is constant for all \( x \).

The accrued benefit function is a year-of-service multiple of this constant

\[ cu\, B_x = \frac{B_r}{r - b} (x - b) = (x - b)k \frac{1}{n} (S_r - S_{r-n}) \]
Benefit functions: Final average

The constant percent modification defines the benefit accrual function $b_x$ as a constant percent of salary, the appropriate percentage is found dividing the projected benefit by the cumulative projected salary

$$ CP \ b_x = \frac{B_r}{S_r} s_x $$

The accrued benefit function is

$$ CP \ B_x = \frac{B_r}{S_r} S_x $$