Course program

• 1. Introduction
• Social insurance, social security and welfare. Insurance and assistance.
• 2. Multiple decrement models (theory of competing risks)
• Populations subject to multiple contingencies. Single decrement tables, central rates of multiple decrement. Multiple decrement tables.
• 3. Demographic projections
• 4. Individual present values and benefit valuation
Course program

5. Funding systems

Present values for 1 (more) cohort(s). Fully funded pension systems: the General Average Premium. Pay as you go pension systems. Terminal funding system. The social security evolution. The three pillars. Mutuality and solidarity.

6. Actuarial liabilities and normal costs


7. Mortality projection and longevity risk


8. Introduction to the actuarial mathematics of health insurance
Text books

Exams and Contacts

• The exams are scheduled in the following dates:
  June 17\textsuperscript{th}, July 22\textsuperscript{nd}, September 23\textsuperscript{rd}

• Web site http://www.unical.it/desf/menzietti/

• Email address massimiliano.menzietti@unical.it

• Students’ reception Friday from 16 to 17
  (channel on TEAMS - Room 12, on the III floor, of Cubo 3C)
Social Insurance, Social Security and Welfare

At first, we need define what these words mean and make a distinction between them.

Some definitions from the web:

- **Social Security**, as a whole, refers to any government system which provides money and benefits to people in need - which usually means people with little or no income.

- **Social Insurance** is a type of such a system, where the people pay into the insurance scheme to mitigate the risk (of unemployment, disability, old age).

- **Welfare** would be another aspect of such a system, where people will receive benefits based on eligibility (need), rather than cause (unemployment).

While social insurances are to a large degree paid for by the participants and often additionally supported by government subsidies, welfare is usually entirely dependent by government funds.
Social Security

Other definitions:

• **Government program** aimed at providing basic needs to citizens who are retired, unemployed, or unemployable due to a disability or disadvantage. It is funded usually by **mandatory payroll contributions** from both the employees and the employers, and from the **government's tax revenue**.

• The principle or practice or a program of public provision (as through social insurance or assistance) for the economic security and social welfare of the individual and his or her family.

• **A U.S. government program** established in 1935 to include old-age and survivors insurance, contributions to state unemployment insurance, and old-age assistance.
Social Insurance

Other definitions:

• Form of compensation provided and controlled by a government for elderly, disable, or unemployed people. Acceptance in a social insurance program is not guaranteed and individuals seeking to be covered have to meet certain requirements.

• Social insurance may come in the form of healthcare or monetary compensation. Medicaid, Medicare, and unemployment compensation are all examples of social insurance programs in the United States.

• Protection of the individual against economic hazards (such as unemployment, old age, or disability) in which the government participates or enforces the participation of employers and affected individuals.
Welfare

Other definitions:

• Government support for the poor and otherwise disadvantaged members of the society, usually through provision of free and/or subsidized goods and services.

• The various social services provided by a state for the benefit of its citizens.
Social Insurance

- **Social insurance** is any government-sponsored program with the following four characteristics:
  - the benefits, eligibility requirements and other aspects of the program are defined by statute;
  - explicit provision is made to account for the income and expenses;
  - it is funded by taxes or premiums paid by (or on behalf of) participants (but additional sources of funding may be provided as well);
  - the program serves a defined population, and participation is either compulsory or heavily subsidized.
Social Security

- Social security is any **government system** that provides monetary assistance to people with an inadequate or no income.
- Social security is enshrined in **Article 22 of the Universal Declaration of Human Rights**, which states: “Everyone, as a member of society, has the right to social security and is entitled to realization, through national effort and international co-operation and in accordance with the organization and resources of each State, of the economic, social and cultural rights indispensable for his dignity and the free development of his personality.”
Social Security

• In its 1952 Social Security (Minimum Standards) Convention (nr. 102), the International Labour Organization (ILO) defined the traditional contingencies covered by social security (9) as including:
  – Survival beyond a prescribed age, to be covered by *old age pensions*;
  – The loss of support suffered by a widowed person or child as the result of the death of the breadwinner (survivor’s benefit);
  – Responsibility for the maintenance of children (family benefit);
  – The treatment of any morbid condition (including pregnancy), whatever its cause (medical care);
  – A suspension of earnings due to pregnancy and confinement and their consequences (maternity benefit);
  – A suspension of earnings due to an inability to obtain suitable employment for protected persons who are capable of, and available for, work (unemployment benefits);
  – A suspension of earnings due to an incapacity for work resulting from a morbid condition (sickness leave benefit);
  – A permanent or persistent inability to engage in any gainful activity (disability benefits);
  – The costs and losses involved in medical care, sickness leave, invalidity and death of the breadwinner due to an occupational accident or disease (employment injuries).
  – People who cannot reach a guaranteed social minimum for other reasons may be eligible for *social assistance* (or welfare, in American English).
• Terminology in this area is somewhat different in the United States from in the rest of the English-speaking world.
  
  – The general term for an action program in support of the well being of poor people in the United States is *welfare program*, and the general term for all such programs is simply *welfare*. In the United States, the term *Social Security* refers to the US social insurance program for all retired and disabled people.
  
  – Elsewhere the term is used in a much broader sense, referring to the economic security society offers when people are faced with certain risks.
Welfare

• **Welfare** is the provision of a minimal level of well-being and social support for citizens and other eligible residents without sufficient current means to support basic needs.

• In most developed countries, welfare is mainly provided by the government from tax revenue, and to a lesser extent by NGOs, charities, informal social groups, religious groups, and inter-governmental organizations.

• The **welfare state** is a concept of government in which the state plays a key role in the protection and promotion of the social and economic well-being of its citizens. It is based on the principles of equality of opportunity, equitable distribution of wealth, and public responsibility for those unable to avail themselves of the minimal provisions for a good life. The general term may cover a variety of forms of economic and social organization.
Definitions synthesis

**Social insurance** is characterized by:
- an Insurance scheme to mitigate the risk (benefits are funded through contribution);
- the program serves a defined population (not all the citizens);
- the benefits are usually limited to old age pension, disability benefit, survivor’s benefit.

**Social security** is characterized by:
- an assistance scheme (partially or totally funded by government's tax revenue);
- the program serves all citizens in need;
- it covers different situations (see ILO list).

**Welfare** is characterized by:
- the fact that people will receive benefits based on eligibility (need, poverty), rather than cause;
- is mainly provided by the government from tax revenue;
- it offers various social services.

*The most part of this course will be devoted to Social Insurance*
Social Insurance vs Private Insurance

The benefits provided by Social Insurance as well as them provided by Social Security are often characterized by some aspects that distinguish them from the typical private insurance studied in actuarial mathematics. Otherwise some aspects are common.

**Similarities**

- In both of them there is pooling of risks;
- they need a specific definitions of the benefits provided;
- as well a specific definitions of eligibility rules and the amount of coverage provided;
- both of them are based on an insurance scheme, so premium (or contribution) are required to meet the expected costs of the benefits.
Social Insurance vs Private Insurance

Differences

• In private insurance a greater emphasis is given on equity between individual while in social insurance programs a greater emphasis on the social adequacy of benefits for all participants.

• Participation in private insurance programs is often voluntary; if the purchase of insurance is mandatory, individuals usually have a choice of insurers. Participation in social insurance programs is generally mandatory; if participation is voluntary, the cost is heavily subsidised enough to ensure essentially universal participation.

• Individually purchased private insurance generally must be fully funded. Social insurance programs are often not fully funded (Most systems of social insurance are funded on an ongoing basis without reference to future liabilities). Social Insurance programs often incorporate solidarity between generations and between “lucky” and “unlucky” people.
Social Insurance vs Private Insurance

Focusing our attention on technical aspects, in social insurance there are situations where benefits are on a group of people instead of a single person (for example a family).

Otherwise an insured can be eligible for benefit not only in case of death/survival but also for other events (disability, unemployment,...).

To deal with the first problem we consider multiple life contracts.

While to deal with the second problem we introduce Multiple-decrement theory.
Multiple-life contracts
Introduction

There are many situations where annuity contracts are sold to a group of several people.

We will concentrate on the case of two lives, the most common arrangement, which is sufficient to exhibit most of the complications.

For example, consider the case of a married couple may wish to purchase an annuity that pays income while either of them is alive.
The joint-life status

Consider a pair of lives, \((x)\) and \((y)\). Suppose that we consider the pair to be in a state of \textit{survival} when \textit{both} of them are alive. In other words, the pair \textit{fails} on the first death. Such a pair is commonly known as a \textit{joint-life status}.

An annuity sold on a joint-life status would provide income as long as the status survived, that is, as long as both individuals were living. Annuity benefits would stop upon the first death.

While it is possible to construct life tables in this case, it is awkward, and it is best to work directly with probabilities. The basic quantity we need is the probability that \textit{both} of \((x)\) and \((y)\) will be alive in \(t\) years. We will denote this probability by \(tP_{xy}\).

We will now derive an expression for \(tP_{xy}\) that can be calculated from the life table for single lives assuming independence between the two lives.

\[
\begin{align*}
  tP_{xy} &= tP_x \cdot tP_y
\end{align*}
\]
The joint-life status

We define

\[ tq_{xy} = 1 - t\rho_{xy} \]

the probability that the joint-life status will fail within \( t \) years, which is the probability that at least one of \((x)\) and \((y)\) will die within \( t \) years.

Note that \( tq_{xy} \) is not equal to \( tq_x \cdot tq_y \).

The latter expression is the probability that both lives will die within \( t \) years, which is certainly less than the probability that at least one will die in this time interval.

To express \( tq_{xy} \) in terms of single-life rates, we have \( tq_{xy} = 1 - \rho_x \cdot \rho_y = 1 - (1 - tq_x) \cdot (1 - tq_y) \), which gives:

\[ tq_{xy} = tq_x + tq_y - tq_x \cdot tq_y \]
The joint-life status

The previous expressions are not quite accurate. They would be perfectly valid if the two lives were completely independent of each other. This is unlikely to be true, however, for two people who would choose to buy an annuity contract together, such as a married couple. One can expect that their lives are intertwined to the extent that life-threatening occurrences for one would also affect the other.

We would expect for the typical buyers of an annuity contract that

\[ t_q_{xy} < t_q_x + t_q_y - t_q_x \cdot t_q_y \]

and

\[ t_p_{xy} > t_p_x \cdot t_p_y \]

For married couples there is an additional factor that supports this result. Statistics bear out the fact that upon the death of one partner, the survivor, deprived of the care and companionship of the deceased, often tends to die earlier than otherwise expected. This has been referred to as the ‘broken heart syndrome’.

For convenience, we will assume independence, unless otherwise stated. However, that most of the key formulas remain true without this assumption. The main reason for this assumption is to conveniently calculate numerical quantities, given the single-life data.
The joint-life annuities

Consider a joint-life annuity contract paying \( c_k \) each year provided that both \( (x) \) and \( (y) \) are alive. In order for a payment to be made at time \( k \) we need both lives to have survived to that time, and the probability of this is just \( k p_{xy} \). The present value of the benefits, denoted by \( \bar{a}_{xy}(c) \), is given by:

\[
\bar{a}_{xy}(c) = \sum_{k=0}^{N-1} c_k v(k) \ k p_{xy}
\]

where \( N = \min\{\omega - x, \omega - y\} \).

In other words, the expression is similar to that of an individual income except for the probabilities employed (\( k p_{xy} \) instead of \( k p_x \)).

In the continuous case we have the formula:

\[
\bar{a}_{xy}(c) = \int_0^N c(t) v(t) \ t p_{xy} dt
\]

for the present value of an annuity, with payments made continuously at the annual rate of \( c(t) \) at time \( t \), provided both \( (x) \) and \( (y) \) are alive.
Last-survivor annuities

Consider a pair of lives (x) and (y), which we consider to be in state of survival if either of them is alive. In other words, the pair fails upon the second death. This is known as a last-survivor status.

The standard symbol for this is $\overline{xy}$, which distinguishes it from a joint-life status. The probability that this status will be in a state of survival at time t is denoted by $tP_{\overline{xy}}$.

To calculate this we use a basic fact from probability theory, already illustrated above, which says that the probability that at least one of two events occurs is given by the sum of the probabilities of each occurring, less the probability that they both occur. This gives:

$$tP_{\overline{xy}} = tP_x + tP_y - tP_{xy}$$

We let $tQ_{\overline{xy}}$ denote the probability that $\overline{xy}$ will fail before time t. Clearly $tQ_{\overline{xy}} = 1 - tP_{\overline{xy}}$. 
Last-survivor annuities

Note that we have:

\[ t q_{xy} = 1 - t p_{xy} = 1 - \left( t p_x + t p_y - t p_{xy} \right) = (1 - t p_x) + (1 - t p_y) - (1 - t p_{xy}) \]

\[ = t q_x + t q_y - t q_{xy} \]

under the assumption of independence

\[ t q_{xy} = t q_x + t q_y - t q_x \cdot t q_y \]

So, if our independence assumption holds, we obtain

\[ t q_{xy} = t q_x \cdot t q_y \]

This can be derived directly by noting that in order for this status to fail within \( t \) years, we need to have both lives fail within \( t \) years.

Formulas for annuities based on the last survivor status are easily obtained. As in the joint-life case we simply replace the subscript \((x)\) by \(( t q_{xy} )\). So, for an annuity paying \( c_k \) at time \( k \), provided that both \((x)\) and \((y)\) are alive, the present value is just:

\[ \ddot{a}_{xy}(c) = \sum_{k=0}^{N-1} c_k v(k) \ k p_{xy} = \ddot{a}_x(c) + \ddot{a}_y(c) - \ddot{a}_{xy}(c) \]

where the upper limit \( M \) now denotes the maximum of \( \omega - x \) and \( \omega - y \).

The analogous formula holds for continuous annuities.
The general two-life annuity contract

There is a demand for two-life annuities that are more flexible than ones we described previously. One popular arrangement is for benefits to continue while either person is alive, but for the amount to reduce when only one of the parties is alive.

For example, a married couple may wish to provide for an annuity at retirement that will pay 60 000 yearly while both are alive, reducing to 40 000 yearly when only one is alive. This reduction reflects the fact that while one person can live more cheaply than two, he/she will need more than half of the amount, since many expenses, such as housing, will not necessarily reduce.

Many variations are possible. Benefits need not be symmetric and could vary according to the particular survivor. A common provision in pension plans is that the income stays level as long as the employee is alive, but will continue to the spouse at a reduced level upon the death of the employee.

We consider a single formula that covers all possibilities.
The general two-life annuity contract

A general life annuity on the pair \((x)\) and \((y)\) can be described by three annuity benefit vectors: \(f\), where \(f_k\) is the amount paid at time \(k\) if only \((x)\) is alive; \(g\), where \(g_k\) is the amount paid at time \(k\) if only \((y)\) is alive; and \(h\), where \(h_k\) is the amount paid at time \(k\) if both \((x)\) and \((y)\) are alive. Let \(j = h - f - g\)

We can think of the contract as three separate annuities. One is a life annuity on \((x)\) with benefit vector \(f\). Another is a life annuity on \((y)\) with benefit vector \(g\). These two will provide for the required payments when only one of the pair is living.

The third contract must adjust the payment when both are alive. If both are living at time \(k\), the first two annuities will provide \(f_k + g_k\), so the third must provide the difference, \(j_k = h_k - f_k - g_k\)

The present value of the complete contract is then the sum of the three separate present values, which is just

\[

d_x(f) + d_y(g) + d_{xy}(j)
\]

This formula reduces the calculation of present values for the general two-life annuity to calculating those for single or joint-life annuities.
The general two-life annuity contract

**Example 1** Verify that \( \ddot{a}_x(f) + \ddot{a}_y(g) + \ddot{a}_{xy}(j) \) reduces to \( \ddot{a}_x(c) + \ddot{a}_y(c) - \ddot{a}_{xy}(c) \), when the yearly benefit is a constant 1 unit.

**Example 2** An annuity on \((x)\) and \((y)\) provides yearly payments as long as either \((x)\) or \((y)\) are alive. Payments begin at 12, but reduce to 8 if \((x)\) only is alive or to 6 if \((y)\) only is alive. Find the present value.

**Example 3** An annuity pays 1 unit each year, provided that at least one of \((50)\) and \((65)\) is living and is over age 70, but not if \((50)\) is alive and under age 65. Find the present value.

**Example 4** Find a formula for the present value of an annuity that pays \(c_k\) at time \(k\), provided that \((y)\) is alive, but \((x)\) is not alive.
The general two-life annuity contract

A contract of the type described in Example 4 is known as a *reversionary annuity*. It provides a life annuity on one life, which does not begin until another life has died. It can be used for a similar purpose as life insurance, except that the proceeds are paid out as a life annuity to another person, rather than as a lump sum.

**Remark** We comment briefly here on the effect of using assumption of independence, in which $tP_{xy}$ was approximated by a quantity that is normally too low.

It follows that this approximation will give values for $\bar{a}_{xy}(b)$ that are slightly too low (assuming the entries of $b$ are nonnegative). Since the coefficient of this term is negative in the most common examples of two-life annuities, as illustrated above, the standard assumption of independence means that most two-life annuity premiums are somewhat higher than they would be if a more realistic model was used.
Multiple-decrement theory
Introduction

In actuarial mathematics usually (but not always) we are concerned with benefits payable upon the occurrence of death. In social insurance, situations where an insured is at the same time subject to several different events (that can have financial impact) are typical.

One example in private insurance is the event of withdrawal or lapse, whereby an insured life terminates the contract and receives a cash value. To properly model a life insurance contract, one must consider two causes of termination, death and withdrawal, operating simultaneously.

Sometimes the insurer must distinguish between different causes of death. For example, some policies have a feature that provides additional death benefits for accidental death as opposed to death from natural causes.

Some policies include disability benefits, providing income for people who can no longer work. For such, the insurer must consider the possibility of disability, as well as death and withdrawal.

For employees inscribed to a pension plan, there are at least four events of interest: disability; death; termination of employment; and retirement.
The basic model

We suppose that we have \( m \) different causes of failure operating simultaneously on a group of lives. These causes are often referred to as decrements, since they bring about a decrease in the number of lives under observation.

The traditional insurance policies involved only one decrement, namely that of death. When there are several decrements we are concerned with what is called in the actuarial literature as multiple-decrement theory, or, in biostatistical contexts, the theory of competing risks.

There is a generalization of these ideas to multi-state insurances and annuities, where individuals can transfer freely between one of several states.
The multiple-decrement table

It is convenient to make use of a generalized life table, known as a **multiple-decrement table**. We will number our causes from 1 to \( m \) and use a superscript \((j)\) to refer to cause \( j \), while we use a superscript \((\tau)\) to refer to the total.

We begin with an arbitrary number \( \ell_0^{(\tau)} \) of lives age 0. We let \( \ell_x^{(\tau)} \) denote the number of individuals from this group who are still surviving at age \( x \). That is, they have not succumbed to any of the \( m \) causes.

We let \( d_x^{(j)} \) denote the number of lives who will fail first from cause \((j)\) between the ages of \( x \) and \( x + 1 \). Let

\[
d_x^{(\tau)} = \sum_{j=1}^{m} d_x^{(j)}
\]

which is the number of people who will fail from some cause between the ages of \( x \) and \( x + 1 \).

It follows that

\[
\ell_{x+1}^{(\tau)} = \ell_x^{(\tau)} - d_x^{(\tau)}
\]

That is, the number of survivors at age \( x + 1 \) is equal to the number of survivors at age \( x \), less those who failed from some cause between the ages of \( x \) and \( x + 1 \).
It is important to note the word ‘first’ used in the definition of $d^{(j)}_x$. The model assumes that any cause of failure results in the individual leaving the group, so they are no longer under observation. For example, if cause 1 denotes death and cause 2 withdrawal, a policyholder who withdraws at age $60_{1/2}$ and then dies at age $60_{3/4}$ would be included in $d^{(2)}_{60}$, but not in $d^{(1)}_{60}$.

Therefore, whenever we refer to failing from a certain cause $j$ in the multiple-decrement model, it is understood that this means that this occurs before failure from any other cause.

We defined a multiple-decrement table starting at age 0, but this could be some other age, depending upon the particular application. For example, the multiple-decrement table for an employee pension plan would begin at the first age the employees become eligible for the plan, perhaps age 25 or so.
Quantities calculated from the multiple-decrement table

We first define probabilities of failure. Let

\[ q_x^{(j)} = \frac{d_x^{(j)}}{\ell_x^{(\tau)}} \]

be the probability that \( x \) will fail first from cause \( j \) within 1 year. In practice, one would start with these probabilities and construct the multiple-decrement table inductively, by calculating

\[ d_x^{(j)} = \ell_x^{(\tau)} q_x^{(j)} \]

Let

\[ k \rho_x^{(\tau)} = \frac{\rho_x^{(\tau)}}{\ell_x^{(\tau)}} \]

be the probability that \( x \) will survive to age \( x + k \) without succumbing to any cause.

\[ n q_x^{(j)} = \sum_{k=0}^{n-1} d_{x+k}^{(j)} \frac{\ell_{x+k}}{\ell_x^{(\tau)}} \]

is the probability that \( x \) will fail as a result of cause \( j \) within \( n \) years.
Quantities calculated from the multiple-decrement table

\[ k p_x^{(\tau)} q_{x+k} = \frac{d^{(j)}_{x+k}}{\ell_x^{(\tau)}} \]

is the probability that \((x)\) will fail from cause \(j\) in the time interval \(k\) to \(k + 1\).

\[ \ell_x^{(j)} = \sum_{k=0}^{\omega-x-1} d^{(j)}_x \]

represents the number of people in our group who will fail from cause \(j\) sometime after age \(x\).

We assume in our model that everyone will eventually fail from some cause (which is obvious if death is one of the causes) so we can write

\[ \ell_x^{(\tau)} = \sum_{j=1}^{m} \ell_x^{(j)} \]

Knowing the value of \(\ell_x^{(j)}\) for all integral values of \(x\) and all \(j\) allows us to complete the table since

\[ d_x^{(j)} = \ell_x^{(j)} - \ell_{x+1}^{(j)} \]
Quantities calculated from the multiple-decrement table

Note that we can write

\[ nq_x^{(j)} = \frac{\ell_x^{(j)} - \ell_x^{(j)}}{\ell_x^{(\tau)}} \]

Another symbol we can define is

\[ np_x^{(j)} = \frac{\ell_x^{(j)}}{\ell_x^{(\tau)}} \]

which is the probability that \((x)\) will fail first from cause \(j\) after time \(n\). Here, instead of thinking of the basic symbol \(np\) as denoting survival to time \(n\), we think of the equivalent formulation as **failing after time** \(n\).
Forces of decrement

**Definition:** For each cause $j$, let

$$\mu^{(j)}(x) = \lim_{h \to 0} \frac{hq_{x}^{(j)}}{h} = \lim_{h \to 0} \frac{\ell_{x}^{(j)} - \ell_{x+h}^{(j)}}{h\ell_{x}^{(\tau)}}$$

the **force of decrement at time** $t$, from cause $(j)$ for $(x)$ is the quantity given by

$$\mu_{x}^{(j)}(t) = \mu^{(j)}(x + t)$$

Note that:

$$\mu_{x}^{(j)}(t) = \lim_{h \to 0} \frac{\ell_{x+t}^{(j)} - \ell_{x+t+h}^{(j)}}{h\ell_{x+t}^{(\tau)}} = \lim_{h \to 0} \frac{\ell_{x+t}^{(j)} - \ell_{x+t+h}^{(j)}}{h\ell_{x+t}^{(\tau)}} = \lim_{h \to 0} \frac{t p_{x}^{(j)} - t + h p_{x}^{(j)}}{h \ell_{x}^{(\tau)}}$$

From which we can obtain:

$$\mu_{x}^{(j)}(t) = -\frac{d}{dt} \frac{tp_{x}^{(j)}}{t p_{x}^{(\tau)}} = \frac{d}{dt} \frac{tq_{x}^{(j)}}{t p_{x}^{(\tau)}}$$
Determining the model from the forces of decrement

The multiple-decrement model is often given from the outset by specifying the forces of decrement, and we must use these to calculate probabilities.

We can define the total force

$$\mu_{x}^{(\tau)}(t) = \sum_{j=1}^{m} \mu_{x}^{(j)}(t) = \lim_{h \to 0} \frac{\ell_{x+t} - \ell_{x+t+h}}{h \ell_{x+t}}$$

So $\mu_{x}^{(\tau)}$ is the same type of quantity as the single-life force of mortality in life insurance, except based on the total decrement rather than just on failure by death.

Obviously we have:

$$\mu_{x}^{(\tau)}(t) = \lim_{h \to 0} \frac{\ell_{x+t}^{(\tau)} - \ell_{x+t+h}^{(\tau)}}{h \ell_{x+t}^{(\tau)}} = \lim_{h \to 0} \frac{\ell_{x+t}^{(\tau)} - \ell_{x+t+h}^{(\tau)}}{h \ell_{x+t}^{(\tau)}} \frac{\ell_{x}^{(\tau)}}{\ell_{x}^{(\tau)}} = -\frac{d}{dt} \frac{tp_{x}^{(\tau)}}{tp_{x}^{(\tau)}}$$
Determining the model from the forces of decrement

Integrating:

\[
\mu_x^{(\tau)}(t) = \frac{-d}{dt} \frac{tp_x^{(\tau)}}{tp_x^{(\tau)}} = \frac{-d}{dt} \left[ \log t p_x^{(j)} \right]
\]

we can deduce that

\[
tp_x^{(\tau)} = e^{-\int_0^t \mu_x^{(\tau)}(r)dr}
\]

From:

\[
\mu_x^{(j)}(t) = \frac{d}{dt} \frac{tq_x^{(j)}}{tp_x^{(\tau)}}
\]

we obtain

\[
s q_x^{(j)} = \int_0^s t p_x^{(\tau)} \mu_x^{(j)}(t) dt
\]
A machine analogy

The difficult part of multiple-decrement theory deals with relationships between the different decrements.

Suppose we have a machine with two components, part 1 and part 2, which work completely independently of each other so that the condition of one part does not affect the operation of the other. In order for the machine to work, both parts must be working, so if either part fails, the machine will fail, even though the other part may be in perfect order. We assume also that both parts cannot fail simultaneously, so we can always identify which part caused the machine to fail.

Suppose we want to compute probabilities of failure over some time period, say a year. We have four quantities of interest. For $j = 1, 2$, let $q'(j)$ be the probability that part $j$ will fail, and let $q(j)$ be the probability that the machine will fail due to the failure of part $j$, meaning that part $j$ failed during the year and was the first of the two parts to fail.
A machine analogy

It is clear from elementary probability theory that

\[ q'(j) \geq q(j) \]

It is also clear that, in general, the two quantities are not equal and we would expect the left hand side to be strictly greater than the right.

Suppose, for example, that sometime during the year part 1 fails, causing the machine to fail, and sometime after that but before the end of the year, part 2 fails. Then the event of the failure of part 2 would have occurred, but not the event of part 2 causing the failure of the machine.

Let \( p'(j) = 1 - q'(j) \) denote the probability that part \( j \) will be working at the end of the year, \( q^{(\tau)} \) denote the probability that the machine will fail within the year, and \( p^{(\tau)} = 1 - q^{(\tau)} \) denote the probability that the machine is working at the end of the year.

The machine can fail in one of two mutually exclusive ways, namely, failure of part 1 or failure of part 2. By elementary probability theory we have

\[ q^{(\tau)} = q^{(1)} + q^{(2)} \]
A machine analogy

In order that the machine be working at the end of the period, both parts must be working. Since the parts work independently, we can multiply probabilities to calculate this, which gives

\[ p^{(\tau)} = p'^{(1)} p'^{(2)} \]

So

\[ q^{(1)} + q^{(2)} = 1 - p'^{(1)} p'^{(2)} \]

which is often written in the form

\[ q^{(1)} + q^{(2)} = q'^{(1)} + q'^{(2)} - q'^{(1)} q'^{(2)} \]

The basic problem of interest is as follows. If we are given the unprimed symbols, can we calculate the primed ones, or if we are given the primed symbols can we calculate the unprimed ones?

In general, we cannot do this uniquely and we will need additional information in order to definitely deduce one set of probabilities from the other.
From $q'(j)$ to $q(j)$

For our particular problem, suppose we are given $q'^{(1)}$ and $q'^{(2)}$ and want to compute the probability that the failure of part 1 caused the machine to fail.

We approximate this by computing the probability that part 1 failed during the year and that at the middle of the year part 2 was still working.

Making an assumption of a uniform distribution of failures for each part, over each year, we have

$$q^{(1)} = q'^{(1)} - \frac{1}{2} q'^{(1)} q'^{(2)}$$

and

$$q^{(2)} = q'^{(2)} - \frac{1}{2} q'^{(1)} q'^{(2)}$$

which clearly satisfy $(q'(j) \geq q(j))$ and $(q^{(1)} + q^{(2)} = q'^{(1)} + q'^{(2)} - q'^{(1)} q'^{(2)})$

Moreover the following holds:

$$q^{(1)} - q^{(2)} = q'^{(1)} - q'^{(2)}$$
From $q^{(j)}$ to $q'^(j)$

To calculate the inverse formula, let $\Delta = q'(1) - q'(2) = q^{(1)} - q^{(2)}$ we have

$$q^{(1)} = q'(1) - \frac{1}{2} (q'(1))^2 + \frac{1}{2} q'(1) \Delta$$

We have a quadratic equation in $q'(1)$ which can be solved

$$q'(1) = \frac{(2 + \Delta) - \sqrt{(2 + \Delta)^2 - 8q^{(1)}}}{2}$$

and then

$$q'(2) = q'(1) - \Delta$$
3 causes of decrements

With 3 causes of decrements, under the assumption of a uniform distribution of failures, we have:

\[ q^{(1)} = q'^{(1)} \left[ 1 - \frac{1}{2} (q'^{(2)} + q'^{(3)}) + \frac{1}{3} q'^{(2)} q'^{(3)} \right] \]

that can be approximated by the following:

\[ q^{(1)} \approx q'^{(1)} \left[ 1 - \frac{1}{2} q'^{(2)} \right] \left[ 1 - \frac{1}{2} q'^{(3)} \right] \]

\[ = q'^{(1)} \left[ 1 - \frac{1}{2} (q'^{(2)} + q'^{(3)}) + \frac{1}{4} q'^{(2)} q'^{(3)} \right] \]
4 causes of decrements

With 4 causes of decrements, under the assumption of a uniform distribution of failures, we have:

\[ q^{(1)} = q'^{(1)} \left[ 1 - \frac{1}{2} \left( q'^{(2)} + q'^{(3)} + q'^{(4)} \right) + \frac{1}{3} \left( q'^{(2)} q'^{(3)} + q'^{(2)} q'^{(4)} + q'^{(3)} q'^{(4)} \right) \right] \]
Associated single-decrement tables

We now return to the original setting dealing with a group of lives, to which we will apply our ‘machine’ model. For a definite example start with a multiple-decrement table with two decrements; cause 1 is death, and cause 2 is disability. Suppose we wish to use this table to construct a regular single-life table relating only to death. Would we be justified in simply taking $q_x^{(1)}$ as the mortality rate for age $(x)$, in the single-life table? The answer is a definite ‘no’.

$q_x^{(1)}$ is not the probability that $(x)$ will die within the year, but rather the probability that $(x)$ will die within the year before becoming disabled.

The actual value of $q_x$ should be larger than $q_x^{(1)}$.

A person could die during the year, after having already left the group by reason of disability. In other words, the mortality rate for age $(x)$ in the single decrement table is analogous to the primed rate in the machine model, and will be denoted by $q'_x^{(1)}$. 
Associated single-decrement tables

We may also want to compute \( q'_x^{(2)} \), the probability that \((x)\) will become disabled during the year, assuming that no other causes of failure are operating. We are computing what these probabilities would be if we could somehow eliminate the possibility of death.

For each cause \( j \), the collection of values \( \{q'_x^{(1)}\} \) for various values of \( x \) is known as the associated single-decrement table for cause \( j \). As we have stressed above, these rates give probabilities of failure for the particular cause \( j \), assuming that no other causes of decrement are operating.

The problem often arises of going from one set of rates to another. We might, for example, construct the multiple-decrement table in the first place by using our knowledge of the associated single-decrement tables. Alternatively, we might have first constructed a multiple table by actually observing the effect of the various causes acting together and wish to use that to construct the associated single-decrement tables. In order to do so we can directly apply the method previously described, provided we make the key assumption that the various causes are acting independently.
Forces of decrement in the associated single-decrement tables

The multiple-decrement model is often constructed from the forces of decrement. A question that naturally arises is, how are these forces of decrement obtained? The answer is that they can be taken as the forces of decrement in the associated single-decrement tables, namely

$$\mu'_x(t) = \frac{-d}{dt} \frac{t'p'_x}{tp'_x}$$

We expect that $q'_x > q_x$ since the person may fail within the year from cause $j$ after first failing from another cause, but this argument does not hold when we are speaking of instantaneous rates of failure rather than failure over a period of positive length.

This by itself is not sufficient for equality but is possible to proof that by virtue of independence

$$\mu'_x(t) = \mu_x(t)$$