Actuarial liabilities and normal costs under different actuarial cost methods

Actuarial liabilities

In the following we consider the definition of different actuarial liabilities under different perspectives, the definitions are given as referred to a single life and only considering old age retirement, so we disregard invalidity and survivor benefits.

Plan Termination Liability (PTL)
The Plan Termination Liability is the present value of the accrued benefits for an active or a pensioner aged \( x \) under the perspective of termination of the pension plan, assuming that retirement starts at age \( r \). In the perspective of plan termination only death excludes benefits. The PTL value for an active (\( x < r \)) is:

\[
(PTL)_x = B_{x_r} - x p_x \ v^{r-x} \bar{a}_r
\]

where \( B_x \) are the accrued benefits as specified by the pension scheme;

\( r - x p_x \) is the survival probability from age \( x \) to age \( r \);

\( v^{r-x} \) is the discount factor for \( r - x \) years;

\( \bar{a}_r \) is the present value of retirement pensions for a retiree of age \( r \):

\[
\bar{a}_r = PVR^p(r) = \sum_{y=r}^{\omega - 1} \frac{p_{xy}^{PP}}{1+i} \left( \frac{1 + \beta}{1 + i} \right)^{y-r}
\]

The PTL value for a pensioner (\( x \geq r \)) is:

\[
(PTL)_x = B_x \bar{a}_x
\]

where \( B_x = B_r (1 + \beta)^{x-r} \) is the amount of pension indexed at age \( x \) and \( B_r \) are the retirement benefits at the age of retirement.
**Plan Continuation Liability (PCL)**

The Plan Continuation Liability is the present value of the accrued benefits for an active or a pensioner aged $x$ under the assumption that the plan will continue to exist (and assuming that retirement starts at age $r$); in the perspective of plan continuation not only death excludes benefits but also withdrawal or disability. The PCL value for an active ($x < r$) is:

$$AB(PCL)_x = B_{x} r - x p_x^{(a)} v^{r - x} a_r$$

where $r p_x^{(a)}$ is the probability to stay in active state from age $x$ to age $r$.

The $AB$ prescript to the plan continuation liability symbol indicates that the liability is based on the accrued benefit as defined by the plan. The following equation holds:

$$AB(PCL)_x = \frac{r - x p_x^{(a)}}{r - x p_x} (PTL)_x$$

The PCL value for a pensioner ($x \geq r$) is obviously equal to the PTL:

$$(PCL)_x = (PTL)_x$$

**Actuarial liabilities under different actuarial cost methods**

First of all let us introduce the present value of the participant's total projected retirement benefit $r(PVFB)_x$ at age $x$ (with $x < r$):

$$r(PVFB)_x = B_{r} r - x p_x^{(a)} v^{r - x} a_r$$

$r(PVFB)_x$ represents the value at age $x$ of the future pension benefits that the participant will receive if she/he remains in the pension plan until age $r$, so is equal to the pension benefit at age $r$ ($B_r a_r$) discounted with the discount factor $v^{r - x}$ and conditioned at the permanence in the pension plan ($r - x p_x^{(a)}$). $r(PVFB)_x$ doesn’t depend on the actuarial cost method adopted.
Several actuarial cost methods are used with pension plans, and each method has an associated actuarial liability. In general terms, a cost method's actuarial liability is equal to the present value of benefits allocated to date, which can be expressed as follows:

\[
(AL)_x = B'_x r - x p_x^{(a)} v^{r-x} \ddot{a}_r
\]

where \( B'_x \) represents the benefits allocated according to one of the actuarial cost methods that will be presented in the following. Observe that, if the benefit function is equal to the accrued benefit as defined by the plan, the expression is identical to the plan continuation liability:

\[
(AL)_x = AB(PCL)_x \text{ if } B'_x = B_x
\]

\((AL)_x\) may also be viewed as the portion of the participant's present value of future benefits, \( r(PVFB)_x \), allocated under the method.

The \((AL)_x\) function is the same as the \( r(PVFB)_x \) function evaluated with \( B'_x \), instead of \( B_r \). Since the actuarial liability represents the proportion of \( r(PVFB)_x \) allocated by the actuarial cost method being used, a generalized actuarial liability definition can be expressed in the following manner:

\[
(AL)_x = k \quad r(PVFB)_x
\]

where \( k \) is a fraction dependent on each cost method, \( k = \frac{B'_x}{B_r} \).
**Actuarial cost methods**

**Accrued Benefit Method**

The actuarial liability under the accrued benefit (AB) method, sometimes referred to as the unit credit method, is equal to the present value of accrued benefits:

\[
AB(AL)_x = B_{x r-x} p_x^{(a)} v^{r-x} \ddot{a}_r = AB(PCL)_x
\]

\[
AB(AL)_x = \frac{B_x}{B_r} r(PVFB)_x
\]

and

\[
AB_k = \frac{B_x}{B_r}
\]

**Benefit Prorate Methods**

There are two benefit prorate methods, generally referred to as *projected unit credit methods*. The actuarial liability under the first version uses the accrued benefit function defined as a service proration of the participant's projected retirement benefit yielding a constant unit benefit allocated to each attained age:

\[
B'_x = CU B_x = \frac{(x - b)}{r - b} B_r
\]

The actuarial liability is defined by the following equation, where the BCU prescript denotes the "Benefit prorate, Constant Unit" version:

\[
BCU(AL)_x = \frac{(x - b)}{r - b} B_{r-x} p_x^{(a)} v^{r-x} \ddot{a}_r
\]

\[
BCU(AL)_x = \frac{(x - b)}{r - b} r(PVFB)_x
\]

and

\[
BCU_k = \frac{(x - b)}{r - b}
\]
The actuarial liability under the second version uses the accrued benefit defined as a salary proration of the participant's projected retirement benefit yielding a benefit allocated to each attained age equal to a constant percent of salary:

\[ B'_x = c_p B_x = B_r \frac{S_x}{S_r} \]

The actuarial liability is defined by the following equation, where the BCP prescript denotes the "Benefit prorate, Constant Percent" version:

\[
BCP(\text{AL})_x = \frac{S_x}{S_r} B_{r,r-x} p_x^{(a)} v^{r-x} \ddot{a}_r
\]

\[
BCP(\text{AL})_x = \frac{S_x}{S_r} r (PVFB)_x
\]

and

\[ BCP_k = \frac{S_x}{S_r} \]

Cost Prorate Methods

There are two cost prorate methods, sometimes referred to as projected benefit cost methods or entry age cost methods. Again the liabilities can be defined in terms of prorated retirement benefits, but in this case the proration is based on temporary employment-based annuities. As we will see later, the pension cost under one version is equal to a constant unit amount throughout the employee's career, whereas the other version has costs equal to a constant percent of the employee's salary.

The actuarial liability under the first version is:

\[
CCU(\text{AL})_x = \frac{\ddot{a}_{b:x-b}^{(a)}}{\ddot{a}_{b:r-b}^{(a)}} B_{r,r-x} p_x^{(a)} v^{r-x} \ddot{a}_r
\]

\[
CCU(\text{AL})_x = \frac{\ddot{a}_{b:x-b}^{(a)}}{\ddot{a}_{b:r-b}^{(a)}} r (PVFB)_x
\]
and

\[ c_{CU} k = \frac{\ddot{a}_{b:x-b}^{(a)}}{\ddot{a}_{b:r-b}^{(a)}} \]

where

\[ \ddot{a}_{x:r}^{(a)} = \sum_{h=0}^{n-1} h p_x^{(a)} v^h \]

The prescript CCU denotes the Cost prorate, Constant Unit version.

The actuarial liability under the second version is:

\[ c_{CP} (AL)_x = \frac{s \ddot{a}_{b:x-b}^{(a)}}{s \ddot{a}_{b:r-b}^{(a)}} B_{r-r-x} p_x^{(a)} v^{r-x} \ddot{a}_r \]

\[ c_{CP} (AL)_x = \frac{s \ddot{a}_{b:x-b}^{(a)}}{s \ddot{a}_{b:r-b}^{(a)}} r (PVFB)_x \]

and

\[ c_{CP} k = \frac{s \ddot{a}_{b:x-b}^{(a)}}{s \ddot{a}_{b:r-b}^{(a)}} \]

where

\[ s \ddot{a}_{x:n}^{(a)} = \sum_{h=0}^{n-1} \frac{s_{x+h}}{s_x} p_x^{(a)} v^h \]

The prescript CCP denotes the Cost prorate, Constant Percent version.

If the salary function is not decreasing with age the following inequalities hold:

\[ 0 \leq \frac{B_x}{B_r} \leq \frac{S_x}{S_r} \leq \frac{x - b}{r - b} \leq \frac{s \ddot{a}_{b:x-b}^{(a)}}{s \ddot{a}_{b:r-b}^{(a)}} \leq \frac{\ddot{a}_{b:x-b}^{(a)}}{\ddot{a}_{b:r-b}^{(a)}} \leq 1 \]
Under all the actuarial cost methods $k$ verify the following condition:

$$k = 0 \text{ when } x = b$$

$k$ increases with age

$$k = 1 \text{ when } x = r.$$ 

Broadly speaking infinite actuarial cost methods are feasible as long as they verify the previous conditions for $k$.

**Normal cost**

Pension costs can be categorized into two fundamental types: *normal costs* and *supplemental costs*. Normal costs represent the annual cost attributed to the current year of service rendered by active participants, with such costs being defined by one of several actuarial cost methods. In theory, the actuarial accumulation of normal costs from entry age to retirement age will be equal to the liability for the employee's pension benefit at retirement. The experience of the plan, however, will not precisely match the underlying actuarial assumptions. Moreover, the plan may have granted benefit credits to years prior to its formation (i.e., for periods when normal costs were not calculated), and/or benefit changes or actuarial assumption changes may occur from time to time. Hence, actual normal costs will not accumulate to the retirement-date liability.

Supplemental costs are designed to resolve the difference between the theoretical and actual accumulation of normal costs, again according to a specified methodology.
Normal costs can be determined on a participant by participant basis, with the plan’s overall costs equal to the sum of each individual’s costs, or they can be determined by a nearly equivalent calculation involving an aggregation of plan participants. These two methodologies suggest another type of classification for actuarial cost methods, namely, individual versus aggregate.

We start with a generalized normal cost function, followed by specific definitions. At this point only the normal cost associated with retirement benefits (based on retirement at age \( r \)) is considered.

**Generalized normal cost function**
The retirement-benefit normal cost (NC) for an employee aged \( x \) can be represented by the following generalized function:

\[
(NC)_x = b' \cdot r - x \cdot p_x^{(a)} \cdot v^{r-x} \cdot \ddot{a}_r
\]

Any normal cost can be specified by the appropriate definition of \( b' \), as described more fully in the following.

In general, normal costs are designed to amortize \( r(PVFB)_b \) over the employee’s working lifetime, where \( b \) is as usual the age of entry and where the pattern of amortization payments is governed by the particular actuarial cost method. Thus, the present value of a participant’s future normal costs at age \( b \), \( r(PVFNC)_b \), is equal to \( r(PVFB)_b \):

\[
r(PVFNC)_b = r(PVFB)_b = B_{r - b} p_b^{(a)} v^{r-b} \ddot{a}_r
\]

where

\[
r(PVFNC)_b = \sum_{h=b}^{r-1} (NC)_{h-h-b} p_h^{(a)} v^{h-b} =
\]
\[
\sum_{h=b}^{r-1} b'_{h} r_{-h} p_{h}^{(a)} v^{r-h} a_{r-h-b} p_{b}^{(a)} v^{h-b}
\]

Considering that
\[
r_{-h} p_{x}^{(a)} h-b p_{b}^{(a)} = r_{-b} p_{b}^{(a)}
\]
and
\[
v^{r-h} v^{h-b} = v^{r-b}
\]

\[
_{r}(PVFNC)_{b} = r_{-b} p_{b}^{(a)} v^{r-b} a_{r} \sum_{h=b}^{r-1} b'_{h}
\]

This value is equal to \(_{r}(PVFB)_{b}\) if \(\sum_{h=b}^{r-1} b'_{h} = B_{r}\) and the technical bases adopted are the same.

This relationship is applicable for the normal costs under all actuarial cost methods, and illustrates that normal costs do indeed amortize \(_{r}(PVFB)_{b}\) over the period from age \(b\) to age \(r\).

Continuing with the amortization concept, it also follows that the actuarial liability at age \(x\) can be obtained as the present value of future benefits (PVFB) at that age less the present value of future normal costs (PVFNC) yet to be made (i.e., the portion of \(_{r}(PVFB)_{x}\) not yet amortized):

\[
_{r}(PVFB)_{x} - _{r}(PVFNC)_{x} =
\]

\[
= B_{r_{r-x} p_{x}^{(a)} v^{r-x} a_{r}} - \sum_{h=x}^{r-1} b'_{h} r_{-h} p_{h}^{(a)} v^{r-h} a_{r-h-x} p_{x}^{(a)} v^{h-x} =
\]

\[
= B_{r_{r-x} p_{x}^{(a)} v^{r-x} a_{r}} - \sum_{h=x}^{r-1} b'_{h} r_{-x} p_{x}^{(a)} v^{r-x} a_{r} =
\]
Another definition of the actuarial liability in terms of normal costs is the so-called retrospective approach, in contrast to the prospective approach studied above. Under this definition the actuarial liability is equal to the accumulated value of past normal costs (AVPNC):

\[
    r(\text{AVPNC})_x = \sum_{h=b}^{x-1} (NC)_h \frac{1}{x-h p_h^{(a)}} (1 + i)^{x-h} = \\
    = \sum_{h=b}^{x-1} b'_{r-h} p_h^{(a)} v^{r-h} \ddot{a}_r \frac{1}{x-h p_h^{(a)}} (1 + i)^{x-h}
\]

Considering that

\[
    r-h p_h^{(a)} \frac{1}{x-h p_h^{(a)}} = r-x p_x^{(a)}
\]

and

\[
    v^{r-h} (1 + i)^{x-h} = v^{r-x}
\]

\[
    r(\text{AVPNC})_x = r-x p_x^{(a)} v^{r-x} \ddot{a}_r \sum_{h=b}^{x-1} b'_{h} = (AL)_x
\]

In theory, normal costs can take on any positive or negative value during an employee's working lifetime. The only theoretical restriction on age-specific normal cost values is that their present value (or accumulated value) satisfy the above relationships.
Since an infinite number of normal cost patterns could be determined such that these conditions hold, there exists an infinite number of possible actuarial cost methods, we only discuss the five related to the actuarial methods previously introduced.

**Normal cost under actuarial cost methods**

**Accrued Benefit Method**

Under this method $b'_x = b_x$ so we have:

$$\text{AB}(NC)_x = b_{x-r}p_x^{(a)}v^{r-x}\ddot{a}_r$$

It can be noticed that first three elements of the equation increase with age, so the accrued benefit method produces a sequence of Normal Cost steeply increasing.

**Benefit Prorate Methods: Constant Unit**

Under this method $b'_x = \text{CU} b_x = \frac{B_r}{r-b} = k \frac{1}{n} (S_r - S_{r-n})$ so we have:

$$\text{BCU}(NC)_x = \frac{B_r}{r-b} \frac{r-x}{P_x^{(a)}v^{r-x}\ddot{a}_r}$$

It can be noticed that the second and the third element of the equation increase with age, so the constant unit version of the benefit prorate method produces a sequence of Normal Cost increasing but less than the accrued benefit method.

**Benefit Prorate Methods: Constant Percent**

Under this method $b'_x = \text{CP} b_x = \frac{B_r}{S_r} s_x$ so we have:

$$\text{BCP}(NC)_x = \frac{B_r}{S_r} s_{x-r-x}p_x^{(a)}v^{r-x}\ddot{a}_r$$

It can be noticed that the second, the third and the forth element of the equation increase with age, so the constant percent version of the benefit prorate method produces a sequence of Normal Cost increasing less than the accrued benefit method but more than the constant unit version of the benefit prorate method.
Cost Prorate Methods: Constant Unit

Under this method \( CCU_k = \frac{B'_{tx}}{B_r} = \frac{\hat{a}_{b:x-b}^{(a)}}{\hat{a}_{b:r-b}^{(a)}} \) so we have:

\[
B'_{tx} = \frac{B_r}{\hat{a}_{b:r-b}^{(a)}} \hat{a}_{b:x-b}^{(a)}
\]

We denote with \( H' \) the ratio \( \frac{B_r}{\hat{a}_{b:r-b}^{(a)}} \) that is constant respect to the age \( x \), \( b'_{tx} \) will be equal to:

\[
b'_{tx} = B'_{tx+1} - B'_{tx} = H' \left( \hat{a}_{b:x+1-b}^{(a)} - \hat{a}_{b:x-b}^{(a)} \right) = H' \left( \sum_{h=b}^{x} h-b P_b^{(a)} v^{h-b} - \sum_{h=b}^{x-1} h-b P_b^{(a)} v^{h-b} \right) = H' \sum_{h=b}^{x} (h-b) P_b^{(a)} v^{h-b}
\]

So the normal cost will be:

\[
CCU(\text{NC})_x = H' \sum_{h=b}^{x} (h-b) P_b^{(a)} v^{h-b} = H' x-b P_b^{(a)} v^{x-b} - x P_x^{(a)} v^{r-x} \hat{a}_r = H' r-b P_b^{(a)} v^{r-b} \hat{a}_r
\]

It can be noticed that all the terms of the equation are constant, so the constant unit version of the cost prorate method produces a sequence of Normal Cost steady.

Cost Prorate Methods: Constant Percent

Under this method \( CCP_k = \frac{B'_{tx}}{B_r} = \frac{s \hat{a}_{b:x-b}^{(a)}}{s \hat{a}_{b:r-b}^{(a)}} \) so we have:

\[
B'_{tx} = \frac{B_r}{s \hat{a}_{b:r-b}^{(a)}} s \hat{a}_{b:x-b}^{(a)}
\]

We denote with \( H'' \) the ratio \( \frac{B_r}{s \hat{a}_{b:r-b}^{(a)}} \) that is constant respect to the age \( x \), \( b'_{tx} \) will be equal to:

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\[ b'_x = B'_{x+1} - B'_x = H'' \left( \frac{s^{(a)}_{b:x+1-b}}{\bar{a}_{b:x+1-b}} - \frac{s^{(a)}_{b:x-b}}{\bar{a}_{b:x-b}} \right) = \]
\[ = H'' \left( \sum_{h=b}^{x} p_{b}^{(a)} v^{h-b} \frac{s_{h}}{s_{b}} - \sum_{h=b}^{x-1} p_{b}^{(a)} v^{h-b} \frac{s_{h}}{s_{b}} \right) = \]
\[ = H'' x-b p_{b}^{(a)} v^{x-b} \frac{s_{x}}{s_{b}} \]

So the normal cost will be:
\[ c_{CCP}^{(NC)}(x) = H'' \frac{s_{x}}{s_{b}} x-b p_{b}^{(a)} v^{x-b} r-x p_{x}^{(a)} v^{r-x} \bar{a}_{r} = H'' r-b p_{b}^{(a)} v^{r-b} \bar{a}_{r} \frac{s_{x}}{s_{b}} \]

It can be noticed that only the last term of the equation depends on age (typically increasing) and is proportional to the salary at age \( x \), so the constant unit version of the cost prorate method produces a sequence of Normal Cost that is a constant percentage of the salary.