Pension Funds and Social Security
Lecture notes

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Week 6
Funding systems

Present values for a cohort of new entrants

Consider a cohort of new entrants in the year $t$ all aged $x$

$$Z(x,t)$$

the **present value of the salaries** they will receive during their active life is given by:

$$S_x^{(t)} = Z(x,t)s(x,t)PVS(x,t)$$

where

$s(x,t)$ is a function that represents the average salary of the whole active population aged $x$ at time $t$;

$PVS(x,t)$ is the individual present value of salaries that is equal to:

$$PVS(x,t) = \sum_{y=x}^{r-1} \frac{s(y,t+y-x)}{s(x,t)} p_x^{aa} v^{y-x}$$

So we have:

$$S_x^{(t)} = Z(x,t) \sum_{y=x}^{r-1} s(y,t+y-x) p_x^{aa} v^{y-x}$$

The **present value of the retirement benefits** for the cohort is:

$$BR_x^{(t)} = Z(x,t)s(x,t)PVR^a(x,t)$$

where $PVR^a(x,t)$ is the individual present value of retirement pensions for an active insured (here is considered only the hypothesis 1) with old-age retirement only at exact age $r$, the extension to the hypothesis 2) is straightforward)

$$PVR^a(x,t) = P(r,s,t+r-x) \frac{s(r,t+r-x)}{s(x,t)} r-x p_x^{aa} v^{r-x} PVR^p(r)$$
Whit $PVR^p(x) = \sum_{y=x}^{\omega-1} y-x p_x^{pp} v_j^{y-x}$, so

$$BR_x(t) = Z(x,t) P(r,s,t+r-x) s(r,t+r-x) r-x p_x^{aa} v^{r-x} PVR^p(r)$$

The **present value of the invalidity benefits** for the cohort is given by:

$$BL_x(t) = Z(x,t) s(x,t) PV^a(x,t)$$

where $PV^a(x,t)$ is the individual present value of invalidity pensions for an active insured:

$$PV^a(x,t) = \sum_{y=x}^{r-1} Pi(y,s,t+y-x) s(y,t+y-x) y-x p_x^{aa} i_y v^{y+1-x} PV^i(y+1)$$

whit $PV^i(x) = \sum_{y=x}^{\omega-1} y-x p_x^{ii} v_j^{y-x}$, so

$$BL_x(t) = Z(x,t) \sum_{y=x}^{r-1} Pi(y,s,t+y-x) s(y,t+y-x) y-x p_x^{aa} i_y v^{y+1-x} PV^i(y+1)$$

The **present value of the survivor benefits** for the cohort will be:

$$BF_x(t) = Z(x,t) s(x,t) \left[ PV^a(x,t) + PV^a:p(x,t) + PV^a:i(x,t) \right]$$

where $PV^a(x,t)$, $PV^a:p(x,t)$ and $PV^a:i(x,t)$ are the individual present value of survivor pensions for an active insured (death in service), the individual present value of survivor pensions for an active insured (death in retirement) and the individual present value of survivor pensions for an active insured (death in disability), respectively.

So the present value for the all the pension benefits will be:
\[ B_x^{(t)} = BR_x^{(t)} + BI_x^{(t)} + BF_x^{(t)} \]
\[ = Z(x, t)s(x, t)[PVR^a(x, t) + PVI^a(x, t) + PVF^a(x, t) + PVF^{a:p}(x, t) + PVF^{a:i}(x, t)] \]

We can compute this value for all the entry ages \( x \) and obtain the present value for the cohort of new entrants in the year \( t \)

\[ S^{(t)} = \sum_x S_x^{(t)} \]
\[ B^{(t)} = \sum_x B_x^{(t)} \]

**Average premium in a fully funded pension system**

In a fully funded pension system the present value of the contributions paid by the active population should be equal to the present value of the pension benefits that the same population will receive in the future.

Contributions are determined multiplying the salary for a contribution rate. If we consider a cohort of new entrants in the year \( t \) all aged \( x \), the present value of the contribution for the cohort is given by:

\[ C_x^{(t)} = c_x^{(t)} S_x^{(t)} \]

where \( c_x^{(t)} \) is the contribution rate applied to the cohort.

The actuarial balance between contributions and benefits can be established for a single cohort of actives taking into account the age differences of new entrants, or for a single cohort applying an average rate regardless of the age of entry, or for more than one cohort applying an average rate equal for all the cohorts.
Contribution rates (premiums) for a cohort of new entrants distinct by age
We can determine the contribution rate (premium) for the new entrants aged \( x \) at the entry time that guarantees the equilibrium between the present value of their contribution and the present value of their future pension benefits:

\[
C_x(t) = c_x(t)S_x(t) = B_x(t)
\]

\[
c_x(t) = \frac{B_x(t)}{S_x(t)} = \frac{Z(x, t)s(x, t)[PVR^a(x, t) + PV1^a(x, t) + PVF^a(x, t) + PVF^{a:p}(x, t) + PVF^{a:i}(x, t)]}{Z(x, t)s(x, t)PVS(x, t)} = \frac{[PVR^a(x, t) + PV1^a(x, t) + PVF^a(x, t) + PVF^{a:p}(x, t) + PVF^{a:i}(x, t)]}{PVS(x, t)}
\]

Average premium for a cohort of new entrants
In a similar way we can determine the average contribution rate (premium) for the cohort of new entrants for all the entry ages \( x \). The present value of the contributions for the cohort is:

\[
C(t) = \sum_x C_x(t) = \sum_x c_x(t)S_x(t)
\]

Assuming the same contribution rate for all the ages \( (c_x(t) = c(t) \forall x) \)

\[
C(t) = \sum_x c(t)S_x(t) = c(t)\sum_x S_x(t) = c(t)S(t)
\]

We require \( C(t) = B(t) \), so:

\[
c(t) = \frac{B(t)}{S(t)}
\]

It follows that:

\[
c(t) = \frac{B(t)}{S(t)} = \frac{\sum_x B_x(t)}{\sum_x S_x(t)}
\]
Remembering that $c_x(t)S_x(t) = B_x(t)$ we have:

$$c(t) = \frac{\sum_x c_x(t)S_x(t)}{\sum_x S_x(t)}$$

So the average premium for the cohort is an average of the premiums for the different entry ages. Observe that usually $c_x(t)$ increases with the age $x$, so the younger new entrants offer solidarity to older ones.

**Average premium for more than a cohort of new entrants – The General Average Premium**

Let now consider more than a cohort (from 1 to $t$) of new entrants, the present value at time 0 of the salaries for all the cohorts is given by:

$$S^{(1:t)} = \sum_{m=1}^{t} S^{(m)} v^m$$

while the present value of the pension benefits for all the cohorts is given by:

$$B^{(1:t)} = \sum_{m=1}^{t} B^{(m)} v^m$$

and the present value of the contributions for all the cohorts is:

$$C^{(1:t)} = \sum_{m=1}^{t} C^{(m)} v^m = \sum_{m=1}^{t} c^{(m)} S^{(m)} v^m$$

Assuming the same contribution rate for all the cohorts ($c^{(m)} = c^{(1:t)} \forall m$)

$$C^{(1:t)} = \sum_{m=1}^{t} c^{(1:t)} S^{(m)} v^m = c^{(1:t)} \sum_{m=1}^{t} S^{(m)} v^m = c^{(1:t)} S^{(1:t)}$$
We can determine the average contribution rate (premium) for the \( t \) cohorts of new entrants that guarantees the equilibrium between the present value of contributions and the present value of pension benefits in this way:

\[
C^{(1:t)} = c^{(1:t)}S^{(1:t)} = B^{(1:t)}
\]

\[
c^{(1:t)} = \frac{B^{(1:t)}}{S^{(1:t)}} = \frac{\sum_{m=1}^{t} B^{(m)} v^m}{\sum_{m=1}^{t} S^{(m)} v^m}
\]

Remembering that:

\[
c^{(m)}S^{(m)} = B^{(m)}
\]

we have:

\[
c^{(1:t)} = \frac{\sum_{m=1}^{t} c^{(m)}S^{(m)} v^m}{\sum_{m=1}^{t} S^{(m)} v^m}
\]

So the average premium for the \( t \) cohorts is given by an average of the premiums for the different cohorts.

Note that we have not considered \( c^{(0)}, S^{(0)} \) and \( B^{(0)} \), that are the amounts for the cohort of active people at the time of evaluation (the initial active population) nor the amounts for people already retired at time 0 (the initial retired population). The first group is characterized by equilibrium contribution rates higher than the future new entrants due to a higher average age and accumulated service. For the second group is impossible to determine a premium on their own salaries do to the fact that they are no longer active, so their benefits should be financed by other ways, e.g. via an initial reserve or by an extra contribution on active population. We neglect this aspect.

If the limits exist:

\[
\lim_{t \to \infty} B^{(1:t)} = \lim_{t \to \infty} \sum_{m=1}^{t} B^{(m)} v^m
\]
\[
\lim_{t \to \infty} S^{(1:t)} = \lim_{t \to \infty} \sum_{m=1}^{t} S^{(m)} v^m
\]

is possible to determine the General Average Premium (GAP):

\[
c_{GAP} = \frac{\sum_{m=1}^{\infty} c^{(m)} S^{(m)} v^m}{\sum_{m=1}^{\infty} S^{(m)} v^m}
\]

Referring to the condition of existence of the above limits, some considerations could be done under simplified assumptions. Let us assume that the number of new entrants evolves over time with a constant rate \( \rho \) and that the distribution by age of new entrants is time invariant so that:

\[
Z(x, t) = Z(x, 0)(1 + \rho)^t
\]

Remembering that the evolution of salaries is given by:

\[
s(x, t) = s(x - 1, t - 1) \frac{ss_x}{ss_{x-1}} (1 + \gamma_I(t))(1 + \gamma_P(t))
\]

where

\( ss_x \) is the age-related salary scale function (merit salary scale) at time \( t_0 \), \( ss_x = s(x, t_0) \);

\( \gamma_I(t) \) is the rate of inflation reflected in the salary increase;

\( \gamma_P(t) \) is the rate of productivity reflected in the salary increase.

Assuming \( b \) as the minimum entry age and \( \gamma_I(t) \) and \( \gamma_P(t) \) constant over time and \( t_0 = 0 \), the following expression holds:

\[
s(x, t) = s(b, t - x + b) \frac{ss_x}{ss_b} [(1 + \gamma_I)(1 + \gamma_P)]^{x-b}
\]

where

\[
s(b, t) = ss_b [(1 + \gamma_I)(1 + \gamma_P)]^t
\]
so we have

\[ s(x, t) = ss_x[(1 + \gamma_I)(1 + \gamma_P)]^t \]

We know that the individual present value of salaries for an active aged \( x \) at time \( t \) is:

\[ PVS(x, t) = \sum_{y=x}^{r-1} \frac{s(y, t + y - x)}{s(x, t)} y-x p_{\alpha\alpha}^a v^{y-x} \]

so we have:

\[ PVS(x, t) = \sum_{y=x}^{r-1} \frac{ss_y[(1 + \gamma_I)(1 + \gamma_P)]^{t+y-x}}{ss_x[(1 + \gamma_I)(1 + \gamma_P)]^t} y-x p_{\alpha\alpha}^a v^{y-x} = \]

\[ = \sum_{y=x}^{r-1} \frac{ss_y}{ss_x} [(1 + \gamma_I)(1 + \gamma_P)]^{y-x} y-x p_{\alpha\alpha}^a v^{y-x} = \]

Recalling that \( v = \frac{1}{1+i} \) and setting \( v_s \) and \( j_s \) so that:

\[ v_s = \frac{1}{1 + j_s} = \frac{(1 + \gamma_I)(1 + \gamma_P)}{1 + i} \]

we obtain:

\[ PVS(x, t) = \sum_{y=x}^{r-1} \frac{ss_y}{ss_x} y-x p_{\alpha\alpha}^a v_s^{y-x} = PVS(x, 0) \]

Therefore the individual present value of salaries for an active aged \( x \) at time \( t \) doesn’t depend on \( t \).

Under the previous assumptions the present value of salaries for a cohort of new entrants aged \( x \) in the year \( t \) is given by:

\[ S_{x}^{(t)} = Z(x, t)s(x, t)PVS(x, t) = \]

\[ = Z(x, 0)(1 + \rho)^t ss_x[(1 + \gamma_I)(1 + \gamma_P)]^t PVS(x, 0) = \]

\[ = S_{x}^{(0)}(1 + \rho)^t [(1 + \gamma_I)(1 + \gamma_P)]^t \]
If we consider the sum for all the ages $x$ we obtain:

$$ S^{(t)} = \sum_x S^{(t)} = \sum_x S^{(0)}(1 + \rho)^t [(1 + \gamma_I)(1 + \gamma_P)]^t $$

$$ = S^{(0)}(1 + \rho)^t [(1 + \gamma_I)(1 + \gamma_P)]^t $$

Now let consider the denominator of the General Average Premium:

$$ \sum_{m=1}^{\infty} S^{(m)} \nu^m = \sum_{m=1}^{\infty} S^{(0)} (1 + \rho)^m \left[ \frac{(1 + \gamma_I)(1 + \gamma_P)}{1 + i} \right]^m = $$

$$ = \sum_{m=1}^{\infty} S^{(0)} (1 + \rho)^m \nu_s^m = S^{(0)} \sum_{m=1}^{\infty} \left( \frac{1 + \rho}{1 + j_s} \right)^m $$

it is immediate to observe that this sum converges if $\left| \frac{1 + \rho}{1 + j_s} \right| < 1$. This condition is verified when $\rho, i, \gamma_I, \gamma_P > -1$ if:

$$(1 + i) > (1 + \rho)(1 + \gamma_I)(1 + \gamma_P)$$

where the product on the right side represents the increase in salary due to demographic evolution, inflation and productivity, respectively; while the left side increases with the interest rate chosen for the evaluation.

A similar demonstration can be performed for the numerator of the General Average Premium.

So under assumptions of time invariance of the evolution of new entrants, of the rate of inflation reflected in the salary increase and of the rate of productivity reflected in the salary increase, it is immediate to verify that is possible to calculate the General Average Premium choosing an adequate interest rate in the evaluation.
Pay As You Go (PAYG) pension systems

In a PAYG pension system the contribution rate is determined in such a way that the sum of the contributions paid in each year is equal to the benefits paid in the same year. Alternatively, the contribution rate could be determined in order to guarantee the equivalence between the contributions paid for a specific number of years and the pension benefits for the same years.

Let us denote with:

\[ S^*(t) \] the salaries function in the year \( t \), which represents the amount of salaries received by the active population in the year \( t \);
\[ c^*(t) \] the contribution rate function in the year \( t \);
\[ C^*(t) \] the contributions function in the year \( t \), with \( C^*(t) = c^*(t)S^*(t) \);
\[ B^*(t) \] the expenditure (benefits) function in the year \( t \), which represents the amount of pension benefits received by the retired population in the year \( t \).

The salary function is obtained in the following way:

\[
S^*(t) = \sum_x \sum_s A(x,s,t)s(x,t) + \frac{1}{2}\sum_x Z(x,t)s(x,t)
\]

The expenditure function \( B^*(t) \) is obtained as sum of the retirement benefits function \( BR^*(t) \), the invalidity benefits function \( BI^*(t) \) and the survivor benefits function \( BF^*(t) \).

The retirement benefits function \( BR^*(t) \) is obtained as:

\[
BR^*(t) = \sum_x R(x,t)\bar{p}(x,t)
\]

where

\( R(x,t) \) is the number of retirement pensioners aged \( x \) in the year \( t \)
\( \bar{p}(x, t) \) is the average retirement pension for retirement pensioners aged \( x \) in the year \( t \)

In a similar way we can obtain \( BL^*(t) \) and \( BF^*(t) \).

In a PAYG system we have for each year \( t \):

\[
C^*(t) = c^*(t)S^*(t) = B^*(t)
\]

so

\[
c^*(t) = \frac{B^*(t)}{S^*(t)}
\]

If we want to define a contribution rate stable over \( t \) years we have:

\[
c^*(1:t) = \frac{B^*(1:t)}{S^*(1:t)}
\]

where

\[
S^*(1:t) = \sum_{m=1}^{t} S^*(m)v^m
\]

\[
B^*(1:t) = \sum_{m=1}^{t} B^*(m)v^m
\]

so

\[
c^*(1:t) = \frac{\sum_{m=1}^{t} B^*(m)v^m}{\sum_{m=1}^{t} S^*(m)v^m}
\]

Remembering that \( c^*(t)S^*(t) = B^*(t) \) we have:

\[
c^*(1:t) = \frac{\sum_{m=1}^{t} c^*(m)S^*(m)v^m}{\sum_{m=1}^{t} S^*(m)v^m}
\]

So the average contribution rate for \( t \) years is obtained as an average of the contribution rate for the different years.
If the limits exist:

\[
\lim_{t \to \infty} B^*(1:t) = \lim_{t \to \infty} \sum_{m=1}^{t} B^*(m)v^m \\
\lim_{t \to \infty} S^*(1:t) = \lim_{t \to \infty} \sum_{m=1}^{t} S^*(m)v^m
\]

Is possible to determine the General Average Contribution Rate (GACR) of a PAYG financial system as:

\[
c_{GACR} = \frac{\sum_{m=1}^{\infty} c^*(m) S^*(m)v^m}{\sum_{m=1}^{\infty} S^*(m)v^m}
\]

It is worth noting that is possible to demonstrate, under quite general assumptions, that \(c_{GAP} = c_{GACR}\).

**Fundamental equation of equilibrium and reserve function**

Any financial system aims at achieving an equilibrium between income and output of the pension scheme, without necessarily equating each contribution to current expenditure (PAYG system) or equating for each cohort the present value of future contributions to the present value of future benefits (fully funded system). In theory an infinite number of financial systems could be envisaged for a pension scheme.

In any case a financial system should verify the following equation (the fundamental equation of equilibrium):

\[
V(t) - V(t-1) = V(t - 1)i + c^*(t)S^*(t) - B^*(t)
\]

where \(V(t)\) is the reserve function in the year \(t\), which represents the excess of inflow over outflow accumulated in the past with interest rate \(i\).

From the fundamental equation of equilibrium, we have:
\[ V(t) = V(t-1)(1+i) + c^*(t)S^*(t) - B^*(t) \]

from which recursively we obtain:

\[
V(t) = V(t-2)(1+i)^2 + c^*(t-1)S^*(t-1)(1+i) + c^*(t)S^*(t) \\
- B^*(t-1)(1+i) - B^*(t)
\]

\[
V(t) = V(t-n)(1+i)^n + \sum_{h=0}^{n-1} c^*(t-h)S^*(t-h)(1+i)^h \\
- \sum_{h=0}^{n-1} B^*(t-h)(1+i)^h
\]

If we assume \( V(0) = 0 \) we have:

\[
V(t) = \sum_{h=0}^{t-1} c^*(t-h)S^*(t-h)(1+i)^h - \sum_{h=0}^{t-1} B^*(t-h)(1+i)^h
\]
equivalently

\[
V(t) = \sum_{h=1}^{t} [c^*(h)S^*(h) - B^*(h)](1+i)^{t-h}
\]

Now let assume that the following equation holds:

\[
\sum_{h=1}^{\infty} c^*(h)S^*(h)(1+i)^{-h} = \sum_{h=1}^{\infty} B^*(h)(1+i)^{-h}
\]

so we assume the equilibrium of the pension scheme on an infinite time horizon. The previous expression can be rewritten as follows:
\[
\sum_{h=1}^{t} c^*(h)S^*(h)(1 + i)^{-h} - \sum_{h=1}^{t} B^*(h)(1 + i)^{-h} = \sum_{h=t+1}^{\infty} B^*(h)(1 + i)^{-h} - \sum_{h=t+1}^{\infty} c^*(h)S^*(h)(1 + i)^{-h}
\]

\[
\sum_{h=1}^{t} [c^*(h)S^*(h) - B^*(h)](1 + i)^{-h} = \sum_{h=t+1}^{\infty} [B^*(h) - c^*(h)S^*(h)](1 + i)^{-(h-t)}
\]

The left side represents the reserve function in the year \(t\), \(V(t)\), valued retrospectively, while the right side represents the difference between the present value of future pension benefits and the present value of future contributions and can be interpreted as a perspective valuation of the reserve function.

Any contribution function \(c^*(t)\) that satisfies the equation:

\[
\sum_{h=1}^{\infty} c^*(h)S^*(h)(1 + i)^{-h} = \sum_{h=1}^{\infty} B^*(h)(1 + i)^{-h}
\]

constitutes a theoretically possible financial system for a pension scheme, however conditions should be imposed in order to exclude \(c^*(t) < 0\) (a reimbursement of contribution) or \(V(t) < 0\) (borrowing to pay current benefits).

The PAYG system is based on the assumption that \(c^*(h)S^*(h) = B^*(h)\ \forall t\), so from the equation of equilibrium we have \(V(t) = 0\ \forall t\).

The GAP instead is based on the concept of a constant contribution rate applicable throughout the subsequent lifetime of the pension scheme, \(c(t) = c\ \forall t\), so:
\[ c = \frac{\sum_{h=1}^{\infty} B^*(h)(1 + i)^{-h}}{\sum_{h=1}^{\infty} S^*(h)(1 + i)^{-h}} = c^{GACR} = c^{GAP} \]

The terminal funding system (TFS)
The Terminal Funding System (TFS) is based on full pre-funding at time of retirement of the future benefits for new pensioners.

Let \( K_a(t) \) represents the capitalized value of the pensions awarded in the year \( t \) (between \( t \) and \( t - 1 \)), the following equality holds (if we exclude the existence of pensioners at time \( t = 0 \)):

\[ \sum_{h=1}^{\infty} B^*(h)(1 + i)^{-h} = \sum_{h=1}^{\infty} K_a(h)(1 + i)^{-h} \]

So the fundamental equation of equilibrium becomes:

\[ \sum_{h=1}^{\infty} c^{TFS}(h)S^*(h)(1 + i)^{-h} = \sum_{h=1}^{\infty} K_a(h)(1 + i)^{-h} \]

A possible (the simplest) solution of the equation is:

\[ K_a(h) = c^{TFS}(h)S^*(h) \quad \forall h \]

from which we obtain:

\[ c^{TFS}(h) = \frac{K_a(h)}{S^*(h)} \]

The reserve function becomes:

\[ V(t) = \sum_{h=1}^{t} [K_a(h) - B^*(h)](1 + i)^{t-h} \]

so the reserve is accumulated only for pensioners.

Some concluding remarks should be made. The PAYG financial system is possible only for public mandatory pension systems, private pension schemes should be based on funding systems that accumulate a reserve that guarantee the payment of the future
benefits to current actives and pensioners. The reserve constitutes a liability for a pension scheme due to the fact that represents the future payments to active participants that are not covered by their future contribution.
Notional Defined Contribution (NDC) systems

The introduction

Traditional pension schemes in social security usually combined in the past a Defined Benefit structure (DB) (the system tells you explicitly your future amount of pension through a formula) with a pay as you go mechanism (PAYG) (the pensions given to the retirees today are funded by the contributions paid by the active people today). This classical architecture aimed at creating an intergenerational solidarity while guaranteeing for retirees a well-defined level of living standard.

The ageing phenomena, caused by a spectacular increase of longevity and a decrease of fertility, has clearly put this kind of arrangements into great difficulties; the fixed level of benefits combined with a sharp decrease of the demographic ratios induce doubts about the long-term financial sustainability of such schemes. Some countries have tried to solve the problem by the way of parametric reforms (increase of the contribution rate, increase of the retirement age, reduction of early retirement programs...). Other ones have chosen to change more dramatically the scheme by introducing a structural reform. Among these strategies, the Notional Defined Contribution mechanism (NDC) has been proposed as a credible alternative.

NDC is still a pay as you go system but instead of using a DB formula to compute the benefit, the scheme mimics a financial Defined Contribution plan (DC): the contributions (based on a fixed contribution rate) are accumulated virtually in an account that receives a virtual return based on notional rates (instead of investment returns in a classical DC). At retirement age, the accumulated virtual account is converted into a lifetime annuity.

All these computations are virtual, just aiming at computing the benefits, because the system stays in pay as you go and contributions are never capitalized but directly used to pay the benefits of present retirees.
NDC system is still a PAYG system, so we have for each year $t$:

$$C^*(t) = c^*(t)S^*(t) = B^*(t)$$

so

$$c^*(t) = \frac{B^*(t)}{S^*(t)}$$

In a NDC system, by definition the contribution rate is fixed: $c^*(t) = c \forall t$.

In a NDC pension system, the pension benefit at retirement depends on the total contributions paid by the individual during his working life. These contributions are virtually accumulated in an individual notional account and converted in an annuity at retirement. The individual account is remunerated each year at a notional rate of return that can be chosen in different ways:

- equal to the growth rate of the total wage in the year $t$, that coincides with the growth rate of contributions. When the steady state is reached by the system, the notional rate is $(1 + \rho)(1 + \gamma_I)(1 + \gamma_P) - 1$; therefore it is affected by both wage and population evolution. This choice guarantees that the NDC system is sustainable in both a steady state and a semi steady state (where the growth rate of actives is constant, but the growth rate of wages could vary). For this reason, such a rate is also known as "natural rate" or "canonical rate".

- equal to the growth of the contribution base per capita that increases at the same rate of the individual wage. This is the rate applied in Sweden.

- as a function of the growth rate of notional GDP. For example, in Italy it is equal to the five-year GDP growth average rate. Note that this notional rate can be equal to the wage bill growth rate only if the distributive shares in GDP remain constant.

In the following, we denote $g(t)$ the notional rate of return in the year $t$. The individual notional account for an individual $i$ aged $x$ at the end of year $t$ evolves as
follows:

\[ m(x; t; i) = [m(x - 1; t - 1; i) + c(x; t; i)] [1 + g(t)] \]

At the end of working life (age \( r \)), the initial amount of pension is determined by dividing the accumulated notional capital to an annuity rate specific for each age \( x \) and time \( t \).

\[ b(r; t; i) = m(r; t; i) \cdot TC_{r,t} \]

where \( TC_{r,t} \) if the transformation coefficient at age \( r \) in the year \( t \), that is equal to:

\[ TC_{r,t} = \frac{1}{PVR^p(r, t) + PVF^p(r, t)} \]

where \( PVR^p(r) \) and \( PVF^p(r) \) are respectively equal to:

\[ PVR^p(r, t) = \sum_{y=r}^{\omega-1} (y-r)p_{r,t}^{pp} \left( \frac{1 + \beta^*}{1 + g^*} \right)^{y-r} \]

\[ PVF^p(r, t) = \sum_{y=r}^{\omega-1} (y-r)p_{r,t}^{pp}q_{y,t}^{pw} \left( \frac{1 + \beta^*}{1 + g^*} \right)^{y+1-x} \theta_F PVF^w(y + 1 + c) \]

Note that \( \beta^* \) and \( g^* \) are the ex-ante pension indexation and the ex-ante notional rate (or interest rate), respectively. Therefore, this evaluation requires an assumption on both the future evolution of pension indexation and notional rate.

This philosophy has been successfully implemented in different European countries (Sweden, Italy, Poland, Latvia...) and is often presented as the definitive solution to ageing.
Undoubtedly, NDC has many important advantages with respect to classical DB schemes. The structure maintains the solidarity principles at the heart of pay as you go and the DC implicit assumption should guarantee the financial stability.

The technique incorporates endogenously different parameters that permit to take into account the demographical evolution; for instance, the notional rate can be based on the real evolution of the wages or of the GDP; the annuity rate is computed for each cohort and can be linked to their own life expectancy.

Despite these appealing characteristics, NDC has in practice some serious drawbacks. If theoretically, the system is equilibrated in a steady state (with constant demographical and economic conditions), the method is not naturally sustainable in dynamic conditions.

Therefore, adjustment mechanisms are necessary to restore the equilibrium. In Sweden for instance, Automatic Balance Mechanisms (ABMs) have been put in place: automatic adjustment rules are applied each year to the notional rate and the indexation rate of pensions, depending on the solvency situation. ABMs are techniques introduced in a public pension scheme in order to correct automatically its parameters; the purpose is to avoid waiting for political interventions.

Different kinds of ABMs have been proposed in the literature for NDC schemes. However, all these methods stay generally in a strict DC environment. All the adjustments correct only the level of present and future benefits and the contribution rate stays unchanged.

From a general point of view, DB and DC are extreme cases with respect to adaptation to external risks: in a pay as you go DB scheme, the contributors bear all the risks by an increase of their contribution, while in a DC scheme, the retirees are the only victims via a decrease of their present and future benefits. This last situation could lead to an impoverishment of the oldest ones.