Pension Funds and Social Security
Lecture notes
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Demographic projections

Introduction to demographic projections
As we have seen previously, in both social insurance and social security the benefits that are typically paid are: old age pensions paid in case of survival beyond a prescribed age; survivors pensions paid to widows/widowers and/or children in case of death of the insured; invalidity (or disability) pensions paid in case of permanent inability to continue the work activity.

In order to measure the pension cost for a cohort of insured the first step is to estimate the numbers of individuals in each of the principal population subgroups: active insureds, old age retirees, invalids, widows/widowers, orphans at discrete time-points starting from given initial values \((t = 0)\).

We denote with

\[A(t)\] the active population at time \(t\);
\[R(t)\] the old age retired population at time \(t\);
\[I(t)\] the invalidity pensioners population at time \(t\);
\[W(t)\] the widows/widowers population at time \(t\);
\[O(t)\] the orphans population at time \(t\).

The demographic projection procedure can be regarded as the iteration of a matrix multiplication operation, typified as follows:

\[n_t = n_{t-1} Q_{t-1}\]

In which \(n_t\) is a row vector whose elements represent the demographic projection values at time \(t\) and \(Q_{t-1}\) is a square matrix of transition probabilities for the interval \((t - 1, t)\), which take the form:

\[n_t = [A(t)\; R(t)\; I(t)\; W(t)\; O(t)]\]
\[
Q_t = \begin{bmatrix}
  p^{aa} & q^{ap} & q^{ai} & q^{aw} & q^{ao} \\
  0 & p^{pp} & 0 & q^{pw} & q^{po} \\
  0 & 0 & p^{ii} & q^{iw} & q^{io} \\
  0 & 0 & 0 & p^{ww} & 0 \\
  0 & 0 & 0 & 0 & p^{oo}
\end{bmatrix}
\]

The elements of the matrix and the symbols have the following significance:

- \(p^{rr}\) denotes the probability of remaining in the same status \(r\);
- \(q^{rs}\) denotes the probability of transition from status \(r\) to status \(s\);
- \(a\), \(p\), \(i\), \(w\), and \(o\) respectively represent active lives, retirees, invalids, widows/widowers and orphans.

The above procedure, however, is not applied at the level of total numbers in the subpopulations. In order to improve precision, each subpopulation is subdivided at least by sex and age.

Preferably, the active population would be further subdivided by past service (or more generally for time spent in the status).

The procedure is applied at the lowest level of subdivision and the results aggregated to give various subtotals and totals.

The matrix \(Q\) will be sex-age specific; it can also be varied over time if required.

**Future entrants**

With regard to future entrants into the scheme three variants are usually considered:

**Variant (a):** The expected total active insured population - by sex and age - in future years is provided exogenously, that is, based on national population or labour force projections.
**Variant (b):** Indications are provided of the expected rate of growth of the total active insured population in each projection year, together with the *relative sex-age* distribution of the corresponding new entrants; often, the same sex-age distribution is assumed for all new entrant generations.

In either case, the actual new entrants of each projection year, by sex and age, would be deduced indirectly.

**Variant (c):** The expected new entrants are known exogenously by sex and age.

New entrants are usually assumed to enter at the middle of the financial year but other assumptions are possible.

**Actuarial basis**

For carrying out the demographic projections it is necessary to adopt an actuarial basis as below described. All the elements should be understood to be sex specific.

For brevity, time is not indicated as a variable, but some or all of the bases may be varied over time.

- The active service table \( \{l^a_x\}, b \leq x \leq r \), where \( b \) is the youngest entry age in insurance and \( r \) the highest retirement age. This is a double decrement table allowing for the decrements of death and invalidity only. The associated rates of decrement are denoted by \( q^a_x \) (mortality) and \( i_x \) (invalidity). Retirement is assumed to take place at exact integral ages, just before each birthday, \( r_x \) denoting the proportion retiring at age \( x \).

- The life table for invalids \( \{l^i_x\}, b \leq x < \omega \) and the associated independent mortality rate \( q^i_x \).

- The life table for retired persons, \( \{l^p_x\}, r^* \leq x < \omega \) (where \( r^* \) is the lowest retirement age), and the associated independent rate of mortality \( q^p_x \).
• The double decrement table for widows/widowers, \( \{l_y^w\} \), \( y^* \leq y < \omega \) (\( y^* \) is the lowest age of a widow/widower), and the associated dependent rates of decrement, \( q_y^w \) (mortality) and \( h_y \) (remarriage).

• The single decrement table for orphans, \( \{l_z^o\} \), \( 0 \leq z < z^* \), where \( z^* \) is the age limit for orphans' pensions and the associated independent rate of decrement \( q_z^o \).

• \( w_x \), the proportion of married persons among those dying at age \( x \).

• \( y_x \), the average age of the spouse of a person dying at age \( x \).

• \( n_x \), the average number of orphans of a person dying at age \( x \).
  
  • \( z_x \), the average age of the above orphans.

• \( \rho(t) \), growth rate for the number of active insured persons in projection year \( t \).

The transition probabilities
The following transition probabilities can be derived:

Active to active
\[
\frac{l_{x+1}^a}{l_x^a} = p_{x}^{aa} = (1 - q_x^a - i_x)(1 - r_{x+1})
\]

Active to retiree
\[
q_{x}^{ap} = (1 - q_x^a - i_x)r_{x+1}
\]

Active to invalid
\[
q_{x}^{ai} = i_x \left(1 - 0.5q_x^i\right)
\]
Active to widow/widower

\[ q_{x}^{aw} = q_{x}^{a}w_{x+0.5}[1 - 0.5(q_{y}^{w} + h_{y})] + i_{x}0.5q_{x}^{i}w_{x+0.75}[1 - 0.25(q_{y}^{w} + h_{y})] \]

Note that equation referring to transition probability from active to widow/widower has two components: the first one relating to deaths of active insured persons in the age range \( \{x, x + 1\} \); and the second one relating to active persons becoming invalid and then dying by age \( x + 1 \).

Expressions for transition probabilities concerning orphans can be derived on the same lines as for widows/widowers.

Retiree to retiree

\[ \frac{l_{x+1}^{p}}{l_{x}^{p}} = p_{x}^{pp} = 1 - q_{x}^{p} \]

Retiree to widow/widower

\[ q_{x}^{pw} = q_{x}^{p}w_{x+0.5}[1 - 0.5(q_{y}^{w} + h_{y})] \]

Invalid to invalid

\[ \frac{l_{x+1}^{i}}{l_{x}^{i}} = p_{x}^{ii} = 1 - q_{x}^{i} \]

Invalid to widow/widower

\[ q_{x}^{iw} = q_{x}^{i}w_{x+0.5}[1 - 0.5(q_{y}^{w} + h_{y})] \]

Widow/widower to widow/widower

\[ \frac{l_{x+1}^{w}}{l_{x}^{w}} = p_{x}^{ww} = 1 - q_{x}^{w}h_{x} \]
Active population (assuming variant (b) for the new entrants)
Starting from the population data on the date of the valuation \( t = 0 \), provided as above indicated, the transition probabilities are applied to successive projections by sex and age (and preferably by past service, in the case of the active population).

In the case of the active population projection, new entrants of the immediately preceding year have to be incorporated before proceeding to the next iteration. The projection formulae for the active insured population are given below.

We denote with \( A(x, s, t) \) the active population aged \( x \) nearest birthday, with curtate past service duration \( s \) years, at time \( t; b \leq x < r, s \geq 0 \), obviously we have:

\[
A(t) = \sum_{x} \sum_{s} A(x, s, t)
\]

We denote with \( N(x, t) \) the new entrants at age \( x \) during the year \((t - 1, t)\) and with \( Z(x, t) \) the active survivors, at time \( t \), of \( N(x, t) \).

We assume that the rate of increase of the total active insured population, \( \rho(t) \), is given.

The total active population at time \( t \) is first projected by the formula:

\[
A(t) = A(t - 1)(1 + \rho(t))
\]

This population can be divided in two groups, insured that were already active the previous year and new entrants:

\[
A(t) = \sum_{b+1 \leq x < r} \sum_{s \geq 1} A(x, s, t) + \sum_{b \leq x < r} Z(x, t)
\]

where

\[
\sum_{b+1 \leq x < r} \sum_{s \geq 1} A(x, s, t) = \sum_{b+1 \leq x < r} \sum_{s \geq 1} A(x - 1, s - 1, t - 1) p_{x-1,s-1}^{aa}
\]

and
\[
\sum_{b \leq x < r} Z(x, t) = \sum_{b \leq x < r} N(x, t) p_{x-0.5,0.5}^{aa}
\]

In this last equation \(p_{x-0.5,0.5}^{aa}\) represents the probability that new entrants during the year \((t - 1, t)\) are still active at time \(t\), and is given by:

\[
p_{x-0.5,0.5}^{aa} = (1 - 0.5(q_{x-1}^a + i_{x-1})) (1 - r_x)
\]

We assume that the *relative sex-age* distribution of the new entrants, \(pr(x)\), is known so that:

\[
N(x, t) = pr(x) N(t)
\]

with \(N(t)\) the total number of new entrants during the year \((t - 1, t)\). \(N(t)\) is given by:

\[
N(t) = \frac{A(t) - \sum_{b+1 \leq x < r} \sum_{s \geq 1} A(x - 1, s - 1, t - 1) p_{x-1,s-1}^{aa}}{\sum_{b \leq x < r} pr(x) p_{x-0.5,0.5}^{aa}}
\]

**Beneficiary population projections**

The projection procedure for the various beneficiary populations is illustrated below with reference to retirement pensioners:

\[
R(x, t) = R(x - 1, t - 1) p_{x-1}^{pp} + \sum_{s} A(x - 1, s, t - 1) q_{x-1,s}^{ap} + N(x, t) q_{x-0.5,0.5}^{ap}
\]

where

\[
q_{x-0.5,0.5}^{ap} = (1 - 0.5(q_{x-1}^a + i_{x-1})) r_x
\]

Similar expressions can be derived for invalidity pensioners and widow/widower.