Individual present values

Introduction
In order to calculate the present values of the contributions paid by an active insured to the pension scheme and the present values of the pension benefits that are paid by the pension scheme to a beneficiary, we need to define the present values for a single life.

Present value of salaries
It is necessary in order to compute the present value of contributions:

$$PVS(x, t) = \sum_{y=x}^{r-1} \frac{s(y, t + y - x)}{s(x, t)} y^{-x} p_x^{aa} v^{y-x},$$

for $b \leq x < r$, where

$$y^{-x} p_x^{aa} = \frac{L_y}{L_x},$$

$$v = \frac{1}{1 + i},$$

and

$s(x, t)$ is a function that represents the average salary of the whole active population aged $x$ at time $t$.

$\{L_x^a\}$ is the survival table for active insured and

$i$ is the interest rate.

Both the two last technical bases are assumed to be time invariant.
Present values of retirement pensions for a retiree

The present value of the retirement pension for a retiree is given by:

\[
PVR^p(x) = \sum_{y=x}^{\omega-1} y-x \cdot p^p (\frac{1 + \beta}{1 + i})^{y-x},
\]

for \(x \geq r\), where

\[y-x \cdot p^p = \frac{L_y^p}{L_x^p},\]

and

\(\beta\) is the pension indexation rate.

\(\{L_x^p\}\) is the survival table for a retiree.

Both the technical bases are assumed to be time invariant.

Setting

\[
\frac{1}{1 + j} = \frac{1 + \beta}{1 + i} \rightarrow j = \frac{1 + i}{1 + \beta} - 1 = \frac{i - \beta}{1 + \beta}
\]

and

\[v_j = \frac{1}{1 + j},\]

we have

\[
PVR^p(x) = \sum_{y=x}^{\omega-1} y-x \cdot p^p \cdot v_j^{y-x}.
\]

Observe that for small values of \(\beta\) is \(j \cong i - \beta\).
Present values of retirement pensions for an active insured

The present values of retirement pensions for active insureds are determined under two different hypotheses:

**hypothesis 1) old-age retirement only at exact age $r$**

$$PVR^a(x, t) = P(r, s, t + r - x) \frac{s(r, t + r - x)}{s(x, t)} r_{x} p^{aa} v^{r-x} PVR^p(r)$$

for $b \leq x < r$, where

$P(r, s, t)$ is a function that represents the retirement pension as a proportion of the final salary, it depends on retirement age $r$, on the computation year $t$ and on the vector of salaries $s$.

**hypothesis 2) old-age retirement only at age $y$ (with $r^* \leq y \leq r$)**

$$PVR^a(x, t) = \sum_{y=r^*}^{r} P(y, s, t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y_{y-x} q^a p_x^{aa} v^{y-x} PVR^p(y)$$

Present values of invalidity pensions

**Present values of invalidity pensions for an invalid**

The present value of a retirement pension for an invalid is given by:

$$PVI^i(x) = \sum_{y=x}^{\omega-1} y_{y-x} p_x^{ii} v^{y-x}$$

for $x \geq b$, where

$$y_{y-x} p_x^{ii} = \frac{L_y^i}{L_x^i}$$

{$L_x^i$} is the survival table for a invalidity pensioners, assumed to be time invariant.
Present values of invalidity pensions for an active insured

The present value of a retirement pension for an active insured is given by:

\[ PV_{I}^{a}(x, t) = \sum_{y=x}^{r-1} P_i(y,s,t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y-x p_{x}^{aa} i_{y} v^{y+1-x} PV_{I}^{i}(y + 1) \]

for \( b \leq x < r \); where

\( P_i(y,s,t) \) is a function that represents the invalidity pension as a proportion of the final salary, it depends on invalidity age \( y \), on the evaluation year \( t \) and on the vector of salaries \( s \), and

\( i_{y} \) is the probability to become disable at age \( y \).

Present values of survivor pensions for the widow/widower (orphans)

Present values of survivor pensions for the widow/widower

The present value of a survivor pension for the widow/widower is given by:

\[ PV_{F}^{w}(x) = \sum_{y=x}^{\omega-1} y-x p_{x}^{ww} v_{j}^{y-x} \]

for \( x \geq 0 \), where

\[ y-x p_{x}^{ww} = \frac{L_{x}^{w}}{L_{x}^{w}} \]

\( \{L_{x}^{w}\} \) is the survival table for a widow/widower, assumed to be time invariant.

Present values of survivor pensions for the orphans

A similar expression can be obtained for an orphan:

\[ PV_{F}^{o}(x) = \sum_{y=x}^{k-1} y-x p_{x}^{oo} v_{j}^{y-x} \]

with \( k \) maximum age at which the benefit is payed to an orphan.
If the family unit is composed by more than one person we use a last survivor annuity formula, for example in the case of a widow aged \( x \) and an orphan aged \( y \) we have:

\[
PVF^{w:o}(x; y) = \theta_w \sum_{t=0}^{\omega-x-1} tP_x^{ww}(1 - tP_y^{oo})v^t + \theta_o \sum_{t=0}^{k-y-1} tP_y^{oo}(1 - tP_x^{ww})v^t + \theta_{w:o} \sum_{t=0}^{k-y-1} tP_x^{ww}tP_y^{oo}v^t,
\]

where

\( \theta_w \) is the proportion of the actual potential pension payed to a spouse alone,

\( \theta_o \) is the proportion of the actual potential pension payed to an orphan alone,

\( \theta_{w:o} \) is the proportion of the actual potential pension payed to a survived family composed by a spouse and an orphan.

Obviously we have:

\[
PVF^{w:o}(x; y) = \theta_w \sum_{t=0}^{\omega-x-1} tP_x^{ww}v^t + \theta_o \sum_{t=0}^{k-y-1} tP_y^{oo}v^t + (\theta_{w:o} - \theta_w - \theta_o) \sum_{t=0}^{k-y-1} tP_x^{ww}tP_y^{oo}v^t.
\]

**Present values of survivor pensions (death in service) for an active insured**

Usually the present value of survivor pensions is determined assuming that only the widow/widower survives adopting an age correction in order to reduce the underestimate produced ignoring orphans:
Present values of survivor pensions (death in retirement)

Present values of survivor pensions (death in retirement) for a retired person

The present value of survivor pensions for a retired person is given by:

\[ PVF^p(x) = \sum_{y=x}^{\omega-1} y_{x}^{pp} q_{y}^{pw} v^{y+1-x} \theta_F PVF^w(y + 1 + c) \]

for \( r \leq x < \omega \), and where

- \( q_{y}^{pw} \) depends on \( q_{y}^{p} \) (the retired death probability) and \( w_y \) (the probability that the retired leaves a family).

A similar expression is obtained for invalidity pensioners.

\[ PVF^i(x) = \sum_{y=x}^{\omega-1} y_{x}^{pi} q_{y}^{iw} v^{y+1-x} \theta_F PVF^w(y + 1 + c) \]
The present value of survivor pensions (death in retirement) for an active insured depends on the hypothesis of the retirement age.

**Present values of survivor pensions (death in retirement) for an active insured:**

**hypothesis 1) old-age retirement only at exact age \( r \)**

\[
PVF_{a:p}^r (x, t) = P(r, s, t + r - x) \frac{s(r, t + r - x)}{s(x, t)} r_{-x}p_x^{aa} v^{r-x} PVF_p^r (r)
\]

for \( b \leq x < r \).

**Present values of survivor pensions (death in retirement) for an active insured:**

**hypothesis 2) old-age retirement only at age \( y \) (with \( r^* \leq y \leq r \))**

\[
PVR_{a:p}^r (x, t) = \sum_{y=r^*}^{r} P(y, s, t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y_{-x}p_x^{aa} q_{y-1}^{ap} v^{y-x} PVF_p^y (y).
\]

**Present values of survivor pensions (death in invalidity retirement) for an active insured**

\[
PVR_{a:i} (x, t) = \sum_{y=x}^{r} P(y, s, t + y - x) \frac{s(y, t + y - x)}{s(x, t)} y_{-x}p_x^{aa} i_{y-1} v^{y-x} PVF_i^y (y)
\]
Salary and benefit functions

Salary function

Previously a function $s(x, t)$ that represents the average salary of the whole active population aged $x$ at time $t$ has been defined.

The classical method for the projection of the insured salary, refers to age- and time-related average salaries which are projected, allowing for the progression of each cohort's average salary according to an age-related salary scale function (or merit salary scale) $ss_x$ and taking into account the escalation of the general level of salaries, but the method assumes an invariant salary scale function.

Although the starting salaries of new entrant cohorts could be varied, this method does not permit the modelling of the variation over time in the age-wise salary structure of the active population. Nor does it allow the salary distribution at each age to be taken into account.

$$s(x, t) = s(x - 1, t - 1) \frac{ss_x}{ss_{x-1}} (1 + \gamma_I(t))(1 + \gamma_P(t))$$

where

$ss_x$ is the age-related salary scale function (merit salary scale);

$\gamma_I(t)$ is the rate of inflation reflected in the salary increase;

$\gamma_P(t)$ is the rate of productivity reflected in the salary increase.

Assuming $b$ as minimum entry age and $\gamma_I(t)$ and $\gamma_P(t)$ constant over time the following expression holds:

$$s(x, t) = s(b, t - x + b) \frac{ss_x}{ss_b} [(1 + \gamma_I)(1 + \gamma_P)]^{x-b}$$

where

$$s(b, t) = ss_b[(1 + \gamma_I)(1 + \gamma_P)]^{t-t_0}.$$
So we have

\[ s(x, t) = ss_x[(1 + \gamma_f)(1 + \gamma_p)]^{t-t_0}. \]

**Benefit functions**

The benefit function is used to determine the amount of benefit paid at retirement, disablement or death.

Let \( b_x \) denote the annual benefit accrual during age \( x \) to age \( x + 1 \) for an age-\( b \) entrant, we refer to as the *benefit accrual function*.

The accrued benefit, denoted by \( B_x \), is equal to the sum of each attained age accrual up to, but not including, age \( x \). This function is called the *accrued benefit function* and is defined by:

\[ B_x = \sum_{t=b}^{x-1} b_t. \]

Obviously \( B_r \) represents the benefit paid at retirement.

We can have different evolution of the benefit functions.

**Benefit functions: Flat unit benefit**

Under a flat unit benefit formula \( b_x \) is equal to a flat annual benefit payable per year of service. Although the attained age subscript is shown here, it is unnecessary since the accrual is independent of age.

The accrued benefit is a years-of-service multiple of the benefit accrual:

\[ B_x = (x - b)b_x. \]

So

\[ B_r = (r - b)b_x. \]
If the final benefit $B_r$ is predefined (given the assumed year of service $r - b$), $b_x$ is equal to

$$\frac{B_r}{(r - b)}.$$ 

**Benefit functions: Career average**

The career average benefit formula has the following definitions for the benefit accrual and the accrued benefit functions at age $x$:

$$b_x = ks_x,$$
$$B_x = kS_x,$$

where $s_x$ is the salary at age $x$ and $S_x$ represents the cumulative salary from entry age $b$ up to, but not including, age $x$:

$$S_x = \sum_{t=b}^{x-1} s_t.$$

The benefit functions under the career average formula follow precisely the pattern of the attained age salary and the cumulative salary.

The benefit at retirement will be

$$B_r = kS_r.$$ 

If we fix the retirement benefit $B_r$ and assume a cumulative salary evolution $S_r$,

$$k = \frac{B_r}{S_r}.$$ 

**Benefit functions: Final average**

The final average benefit formula is somewhat more complicated. Let $n$ denote the number of years over which the active’s salary prior to retirement is to be averaged, and let $k$ equal the proportion of the average salary provided for year of service. The
projected retirement benefit, assuming retirement occurs at the beginning of age \( r \), is defined as:

\[
B_r = k(r - b) \frac{1}{n} \sum_{t=r-n}^{r-1} s_t = k(r - b) \frac{1}{n} (S_r - S_{r-n}).
\]

The attained age benefit accrual and the accrued benefit functions can be defined in several ways under this benefit formula.

One approach is to define \( B_x \) according to the benefit formula based on the participant’s current salary average:

\[
B_x = k(x - b) \frac{1}{n} (S_x - S_{x-n}).
\]

Where \( n \) is the smaller between the years specified in the benefit formula or \( x - b \).

The corresponding benefit accrual at age \( x \) can be determined by the following:

\[
b_x = B_{x+1} - B_x
= k \frac{1}{n} (S_{x+1} - S_{x+1-n}) + k(x - b) \frac{1}{n} [(S_{x+1} - S_{x+1-n}) - (S_x - S_{x-n})]
= k \frac{1}{n} (S_{x+1} - S_{x+1-n}) + k(x - b) \frac{1}{n} [s_x - s_{x-n}].
\]

The first term is the portion of the benefit earned in the current year based on the participant’s current \( n \)-year salary average, while the second term represents the portion earned (loosed) as a result of the increase (decrease) in the \( n \)-year salary average base.

The increase in salary base \( \frac{1}{n} [s_x - s_{x-n}] \) is multiplied by years-of-service to date. Therefore, the benefit accrual during age \( x \) includes an implicit updating of previous accruals which can cause \( b_x \) to be a steeply increasing function of \( x \).

As we will see later, steeply increasing benefit accrual and accrued benefit functions produce even steeper pension cost function under some methods of determining
pension costs and liabilities, characteristics that may be undesirable. Consequently, two modifications to benefit functions that are based on the plan’s formula have been developed in order to mitigate this effect: the constant unit modification and the constant percent modification.

The constant unit modification defines the benefit accrual function as a pro rata share of the participant’s retirement-age projected benefit:

\[ cu b_x = \frac{B_r}{r - b} = k \frac{1}{n} (S_r - S_{r-n}), \]

that is constant for all \( x \).

The accrued benefit function is a year-of-service multiple of this constant:

\[ cu B_x = \frac{B_r}{r - b}(x - b) = (x - b) k \frac{1}{n} (S_r - S_{r-n}). \]

The constant percent modification defines the benefit accrual function \( b_x \) as a constant percent of salary, the appropriate percentage is found dividing the projected benefit by the cumulative projected salary:

\[ cp b_x = \frac{B_r}{S_r} s_x. \]

The accrued benefit function is:

\[ cp B_x = \frac{B_r}{S_r} S_x. \]