Longevity risk

Definition of longevity risk

Mortality risk may emerge in different ways. Three cases can in particular be envisaged.

1. One individual may live longer or less than the average lifetime in the population to which she/he belongs. In terms of the frequency of deaths in the population, this may result in observed mortality rates higher than expected in some years, lower than expected in others, with no apparent trend in such deviations.

2. The average lifetime of a population may be different from what is expected. In terms of the frequency of deaths, it turns out that mortality rates observed in time in the population are systematically above or below those coming from the relevant mortality table.

3. Mortality rates in a population may experience sudden jumps, due to critical living conditions, such as influenza epidemics, severe climatic conditions (e.g. hot summers), natural disasters and so on.

In all the three cases, deviations in mortality rates with respect to what is expected are experienced; an illustration is sketched in the following figure where, with reference to a given cohort, in each panel dots represent mortality rates observed along time, whereas the solid line plots their forecasted level.
Case (1) is the well-known situation of possible deviations around expected mortality rates; the mortality risk here comes out as a risk of random fluctuations, which is traditional in the insurance business, in both the life and the non-life area (actually, it is the basic grounds of the insurance business). It concerns the individual position, and as such its severity reduces as the single position becomes negligible in respect of the overall portfolio. This risk can be hedged through the realization of a proper pooling effect, since it reduces as soon as the portfolio is made of similar policies and its size is large enough, as well as through traditional risk transfer arrangements.

Under case (2), deviations are from expected values, rather than around them, hence their systematic nature. This may be the result of either a misspecification of the relevant mortality model (e.g. because the time-pattern of actual mortality differs from that implied by the adopted mortality table) or a biased assessment of the relevant parameters (e.g. due to a lack of data). The former aspect is referred to as the model risk, the latter as the parameter risk. The term uncertainty risk is often used to refer to model and parameter risk jointly, meaning uncertainty in the representation of a phenomenon (e.g. future mortality). When adult-old ages are concerned, uncertainty risk may emerge in particular because of an unanticipated reduction in mortality rates (as is presented in the mid-panel of the figure, where the mortality profile of the cohort is better captured by the dashed line rather than by the solid line). In this case, the term longevity risk is used instead of uncertainty risk. It must be stressed that longevity risk concerns aggregate mortality; so pooling arguments do not apply for its hedging.

In case (3), a catastrophe risk emerges, namely the risk of a sudden and short-term rise in the frequency of deaths. Similar to case (2), aggregate mortality is concerned; however, when compared with longevity risk, the time-span involved by the emergence of the risk needs to be stressed: short term in case (3), long-term (possibly,
permanent) in case (2). Clearly, a proper hedging of catastrophe risk is required when
death benefits are dealt with (whilst when dealing with life annuities, profit arises
because of the higher mortality experienced). The usual pooling arguments do not
apply; however, diversification effects may be realized and risk transfers can be
conceived as well.

**Measuring longevity risk in a static framework**

In this section we highlight the impact of longevity risk. With reference to a portfolio
comprising one cohort of annuitants (or a cohort of pensioners, in either case a
homogeneous group is considered), the distribution of both the present value of
future payments and annual outflows is investigated. The provider could be an insurer
or a pension fund.

A static representation is considered for evolving mortality and, in particular,
parameter risk is addressed. To understand better the impact of longevity risk, a
comparison is made with process risk.

We assume:

\[
\frac{q_x(t)}{p_x(t)} = G(\tau) (K(\tau))^x
\]

where \( \tau = t - x \) is the year of birth of the cohort. This is the third term of the first
Heligman–Pollard law, that is, the one describing the old-age pattern of mortality.
Note, in particular, that the relevant parameters are cohort-specific.

Whilst the age-pattern of mortality for cohort \( \tau \) is accepted to be logistic, namely:

\[
q_x(t) = \frac{G(\tau) (K(\tau))^x}{1 + G(\tau) (K(\tau))^x}
\]

Uncertainty concerns the level of parameters \( G(\tau), K(\tau) \).
We define five alternative sets of parameters, quoted in following table which also shows the expected lifetime ($\mathbb{E}_{T_{65}|A_h(\tau)}$) and the standard deviation ($\sqrt{\text{Var}_{T_{65}|A_h(\tau)}}$) of the lifetime at age 65 conditional on a given set of parameters.

<table>
<thead>
<tr>
<th>$G(\tau)$</th>
<th>$A_1(\tau)$</th>
<th>$A_2(\tau)$</th>
<th>$A_3(\tau)$</th>
<th>$A_4(\tau)$</th>
<th>$A_5(\tau)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K(\tau)$</td>
<td>6.378E-07</td>
<td>3.803E-06</td>
<td>2.005E-06</td>
<td>1.060E-06</td>
<td>3.149E-06</td>
</tr>
<tr>
<td></td>
<td>1.14992</td>
<td>1.12347</td>
<td>1.13025</td>
<td>1.13705</td>
<td>1.11962</td>
</tr>
<tr>
<td>$\mathbb{E}<em>{T</em>{65}</td>
<td>A_h(\tau)}$</td>
<td>20.170</td>
<td>20.743</td>
<td>21.849</td>
<td>22.887</td>
</tr>
<tr>
<td>$\sqrt{\text{Var}<em>{T</em>{65}</td>
<td>A_h(\tau)}}$</td>
<td>7.796</td>
<td>8.780</td>
<td>8.707</td>
<td>8.602</td>
</tr>
</tbody>
</table>

It emerges that, in terms of the survival function itself, the alternative assumptions imply different levels of rectangularization (i.e. squaring of the survival function, as it is witnessed by $\sqrt{\text{Var}_{T_{65}|A_h(\tau)}}$) and expansion (i.e. forward shift of the adult age at which most deaths occur, which is then reflected in the value of $\mathbb{E}_{T_{65}|A_h(\tau)}$).

Assumption $A_3(\tau)$ will be referred to as the best-estimate description of the mortality trend for cohort $\tau$.

The portfolio we refer to consists of one cohort of immediate life annuities. We assume that all annuitants are aged $x_0$ at the time $t_0$ of issue. To shorten the notation, time $t$ will be recorded as the time elapsed since the policy issue, that is, it is the policy duration; hence, at policy duration $t$ the underlying calendar year is $t_0 + t$. The lifetimes of annuitants are assumed, conditional on any given survival function, to be independent of each other and identically distributed. Since our objective is the measurement of longevity risk only, we disregard uncertainty in financial markets; hence, a given flat yield curve is considered. All of the annuitants are entitled to a fixed annual amount.

Let $N_t$ be the random number of annuitants at time $t$, $t = 0, 1, \ldots$, with $N_0$ a specified number (viz the initial size of the portfolio). Whenever the current size of the portfolio is an observed quantity, we will denote it as $n_t$; so $N_0 = n_0$. Quantities relating to the
The in-force portfolio at policy time $t$ is defined as

$$
\Pi_t = \{j | T_x^{(j)} > t \}
$$

Quantities relating to the portfolio are then labelled with $(\Pi)$ on the top. Annual outflows for the portfolio are defined, for $t = 1, 2, \ldots$, as

$$
B_t^{(\Pi)} = \sum_{j:j \in \Pi_t} b^{(j)}
$$

where $b_{(j)}$ is the annual amount to annuitant $j$. The present value of future payments at time $t$, $t = 0, 1, \ldots$, may at first be defined for one annuitant as

$$
Y_t^{(j)} = b^{(j)} a_{K_x^{(j)}}
$$

By summing up in respect of in-force policies, we obtain the present value of future payments for the portfolio:

$$
Y_t^{(\Pi)} = \sum_{j:j \in \Pi_t} Y_t^{(j)}
$$

We are interested in investigating some typical values of $B_t^{(\Pi)}$ and $Y_t^{(\Pi)}$, as well as the coefficient of variation and some percentiles. We will in particular consider the impact of longevity risk in relation to the size of the portfolio. So, unless otherwise stated, a homogeneous portfolio in respect of annual amounts is considered; that is, we set $b_{(j)} = b$ for all $j$. Note that in this case we have:

$$
B_t^{(\Pi)} = b N_t
$$

whilst the present value of future payments for the portfolio may also be expressed as:
where $i$ is the annual interest rate. For a homogeneous portfolio, in the following $Y_t^{(1)}$ is used to denote the present value of future payments to a generic annuitant.

We first adopt an approach where all valuations are then conditional on a given mortality assumption. We have

$$\mathbb{E}[Y_t^{(\Pi)}|A_h(\tau), n_t] = n_t \mathbb{E}[Y_t^{(1)}|A_h(\tau)]$$

Because we are assuming independence of the annuitants’ lifetimes, conditional on a given mortality trend, the following results hold:

$$\text{Var}[Y_t^{(\Pi)}|A_h(\tau), n_t] = n_t \text{Var}[Y_t^{(1)}|A_h(\tau)]$$

$$\text{CV}[Y_t^{(\Pi)}|A_h(\tau), n_t] = \frac{1}{\sqrt{n_t}} \frac{\sqrt{\text{Var}[Y_t^{(1)}|A_h(\tau)]}}{\mathbb{E}[Y_t^{(1)}|A_h(\tau)]}$$

where $n_t$ is the size of the in-force portfolio, observed at the valuation policy time $t$.

The coefficient of variation, in particular, allows us to investigate the effect of the size of the portfolio on the overall riskiness. Expression of CV shows that, in relative terms, the riskiness of the portfolio decreases as $n_t$ increases. Thus, we have

$$\lim_{n_t \to \infty} \text{CV}[Y_t^{(\Pi)}|A_h(\tau), n_t] = 0$$

This represents the well-known result that the larger is the portfolio, the less risky it is, since with high probability the observed values will be close to the expected ones.

In the following tables, we provide an example, in which the age at entry is $x_0 = 65$, the interest rate is 3% p.a., the annual amount of each annuity is $b(j)=1$. It then follows that $B_t^{(\Pi)} = N_t$. 7
Table 7.4 shows the coefficient of variation of the present value of future payments, conditional on the best-estimate scenario: $\text{CV}[Y_t^{(\Pi)}|A_3(\tau), n_t]$.

<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$n_0 = 1$</th>
<th>$n_0 = 100$</th>
<th>$n_0 = 1000$</th>
<th>$n_0 = 10,000$</th>
<th>...</th>
<th>$n_0 \rightarrow \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>31.974%</td>
<td>2.982%</td>
<td>0.969%</td>
<td>0.027%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>5</td>
<td>38.514%</td>
<td>3.618%</td>
<td>1.156%</td>
<td>0.030%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>10</td>
<td>47.039%</td>
<td>4.452%</td>
<td>1.397%</td>
<td>0.038%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>15</td>
<td>58.973%</td>
<td>5.626%</td>
<td>1.734%</td>
<td>0.056%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>20</td>
<td>77.647%</td>
<td>7.469%</td>
<td>2.259%</td>
<td>0.103%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>25</td>
<td>111.894%</td>
<td>10.853%</td>
<td>3.218%</td>
<td>0.250%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>30</td>
<td>189.580%</td>
<td>18.541%</td>
<td>5.379%</td>
<td>0.883%</td>
<td>...</td>
<td>0%</td>
</tr>
<tr>
<td>35</td>
<td>424.200%</td>
<td>41.832%</td>
<td>11.815%</td>
<td>5.202%</td>
<td>...</td>
<td>0%</td>
</tr>
</tbody>
</table>

Table 7.4 shows the coefficient of variation of the present value of future payments for some initial sizes of the portfolio, per unit of expected value. Only the best-estimate assumption is considered. As far as the coefficient of variation is concerned, we note that at any given time it decreases rapidly as the size of the portfolio increases, as we commented on earlier. For a given initial portfolio size, the coefficient of variation increases in time; this is due to the decreasing residual size of the portfolio and to annuitants becoming older as well.

We now assign a probability distribution ($\rho$) on the set $A(\tau)$. The unknown mortality trend, assumed to lie in $A(\tau)$, is denoted by $\tilde{A}(\tau)$. For the unconditional expected present value of future payments, the following relations hold:

$$
\mathbb{E}[Y_t^{(\Pi)}|n_t] = \mathbb{E}_\rho[\mathbb{E}[Y_t^{(\Pi)}|\tilde{A}(\tau), n_t]] = n_t \mathbb{E}_\rho[\mathbb{E}[Y_t^{(1)}|\tilde{A}(\tau)]]
$$

$$
= n_t \sum_{h=1}^{m} \mathbb{E}[Y_t^{(1)}|A_h(\tau)] \rho_h = n_t \mathbb{E}[Y_t^{(1)}]
$$

Where $\mathbb{E}[Y_t^{(1)}] = \sum_{h=1}^{m} \mathbb{E}[Y_t^{(1)}|A_h(\tau)] \rho_h$. The unconditional variance of $Y_t^{(\Pi)}$ can be calculated as
\[
\text{Var}[Y_t^{(\Pi)}|n_t] = \mathbb{E}_\rho[\text{Var}[Y_t^{(\Pi)}|\tilde{A}(\tau), n_t]] + \text{Var}_\rho[\mathbb{E}[Y_t^{(\Pi)}|\tilde{A}(\tau), n_t]] \\
= n_t \mathbb{E}_\rho[\text{Var}[Y_t^{(1)}|\tilde{A}(\tau)]] + n_t^2 \text{Var}_\rho[\mathbb{E}[Y_t^{(1)}|\tilde{A}(\tau)]] \\
= n_t \sum_{b=1}^{m} \text{Var}[Y_t^{(1)}|A_b(\tau)] \rho_b \\
+ n_t^2 \sum_{b=1}^{m} \left(\mathbb{E}[Y_t^{(1)}|A_b(\tau)] - \mathbb{E}[Y_t^{(1)}]\right)^2 \rho_b 
\]

The first term in the expression for the variance reflects deviations around the expected value; so it can be thought of as a measure of process risk. The second term, instead, reflects deviations from the expected value (i.e. systematic deviations) and so it may be thought of as a measure of longevity (namely parameter, in our example) risk. Under the unconditional valuation, the coefficient of variation now takes the following expression:

\[
\text{CV}[Y_t^{(\Pi)}|n_t] = \frac{\sqrt{\text{Var}[Y_t^{(\Pi)}]}}{\mathbb{E}[Y_t^{(\Pi)}]} \\
= \sqrt{\frac{1}{n_t} \mathbb{E}_\rho[\text{Var}[Y_t^{(1)}|\tilde{A}(\tau)]] + \text{Var}_\rho[\mathbb{E}[Y_t^{(1)}|\tilde{A}(\tau)]]} \\
= \sqrt{\mathbb{E}^2[Y_t^{(1)}]} \\
= \sqrt{\mathbb{E}^2[Y_t^{(1)}]} 
\]

The first term under the square root shows that random fluctuations represent a pooling risk, since (in relative terms) their effect is absorbed by the size of the portfolio. This result is similar to that obtained under the valuation conditional on a given mortality trend. The second term, instead, shows that systematic deviations constitute a non-pooling risk, which is not affected by changes in the portfolio size. In particular, the asymptotic value of the risk index
can be thought of as a measure of that part of the mortality risk which is not affected by simply changing the size of the portfolio.

We now describe a numerical example of the results presented above. We consider the same inputs of the previous Example. We assign to $A(\tau)$ the following weights

![Table 7.8. Probability distribution on $A(\tau)$](image)

<table>
<thead>
<tr>
<th>Assumption</th>
<th>Weight $\rho_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1(\tau)$</td>
<td>0.1</td>
</tr>
<tr>
<td>$A_2(\tau)$</td>
<td>0.1</td>
</tr>
<tr>
<td>$A_3(\tau)$</td>
<td>0.6</td>
</tr>
<tr>
<td>$A_4(\tau)$</td>
<td>0.1</td>
</tr>
<tr>
<td>$A_5(\tau)$</td>
<td>0.1</td>
</tr>
</tbody>
</table>

the best-estimate assumption ($A_3(\tau)$) has been given the highest weight. The residual weight has been spread out uniformly on the remaining assumptions.

Table 7.9 shows the unconditional expected value of future payments. Its magnitude is driven by the best-estimate assumption, as seen by comparison with the results in Table 7.2.
In Table 7.10, the unconditional variance of $Y_t^{(Π)}$ for some portfolio sizes is shown, split into the pooling and non-pooling components. For comparison with the conditional valuation, also the case $n_0 = 1$ is quoted. We note the increase in the magnitude of the variance, due to the nonpooling part, as the portfolio size increases. Whenever the portfolio is large at policy issue, the non-pooling component remains important relative to the pooling component even at high policy durations.
The behaviour of the coefficient of variation in respect of the portfolio size is illustrated in Table 7.11. When compared with the case allowing for process risk only (see Table 7.4), the risk index decreases more slowly as the portfolio size increases. We note, in particular, its positive limiting value, which is evidence of the magnitude of the systematic risk.

<p>| Time | Variance of $Y^{(1)}<em>{t|n_3}$ &amp; $Y^{(2)}</em>{t|n_1}$ | Pooling part $E_{\tilde{y}}[\text{Var}(\tilde{y}<em>{t|n_3})|\tilde{y}</em>{t|n_1}]$ | Non-pooling part $\text{Var}<em>{\tilde{y}}[\text{Var}(\tilde{y}</em>{t|n_3})|\tilde{y}_{t|n_1}]$ |
|------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| 0    | 23.916                                         | 98.90%                                          | 1.10%                                           |
| 5    | 23.303                                         | 98.73%                                          | 1.27%                                           |
| 10   | 20.369                                         | 98.56%                                          | 1.44%                                           |
| 15   | 15.322                                         | 98.43%                                          | 1.57%                                           |
| 20   | 9.331                                          | 98.45%                                          | 1.55%                                           |
| 25   | 4.202                                          | 98.75%                                          | 1.25%                                           |
| 30   | 1.221                                          | 99.30%                                          | 0.70%                                           |
| 35   | 0.187                                          | 99.79%                                          | 0.21%                                           |
| Initial portfolio size: $n_3 = 100$ | | | |
| 0    | 50.026                                         | 47.28%                                          | 52.72%                                          |
| 5    | 52.493                                         | 43.83%                                          | 56.17%                                          |
| 10   | 49.436                                         | 40.61%                                          | 59.39%                                          |
| 15   | 39.209                                         | 38.46%                                          | 61.54%                                          |
| 20   | 23.670                                         | 38.81%                                          | 61.19%                                          |
| 25   | 9.391                                          | 44.18%                                          | 55.82%                                          |
| 30   | 2.062                                          | 58.81%                                          | 41.19%                                          |
| 35   | 0.226                                          | 82.60%                                          | 17.40%                                          |
| Initial portfolio size: $n_3 = 1,000$ | | | |
| 0    | 287.390                                        | 8.23%                                           | 91.77%                                          |
| 5    | 317.858                                        | 7.24%                                           | 92.76%                                          |
| 10   | 313.680                                        | 6.40%                                           | 93.60%                                          |
| 15   | 256.365                                        | 5.88%                                           | 94.12%                                          |
| 20   | 154.023                                        | 5.96%                                           | 94.04%                                          |
| 25   | 56.368                                         | 7.33%                                           | 92.67%                                          |
| 30   | 9.707                                          | 12.49%                                          | 87.51%                                          |
| 35   | 0.580                                          | 32.20%                                          | 67.80%                                          |
| Initial portfolio size: $n_3 = 10,000$ | | | |
| 0    | 2,661.023                                      | 0.89%                                           | 99.11%                                          |
| 5    | 2,971.308                                      | 0.77%                                           | 99.23%                                          |
| 10   | 2,956.118                                      | 0.68%                                           | 99.32%                                          |
| 15   | 2,427.922                                      | 0.62%                                           | 99.38%                                          |
| 20   | 1,457.548                                      | 0.63%                                           | 99.37%                                          |
| 25   | 528.334                                        | 0.79%                                           | 99.21%                                          |
| 30   | 86.159                                         | 1.41%                                           | 98.59%                                          |
| 35   | 4.120                                          | 4.53%                                           | 95.47%                                          |</p>
<table>
<thead>
<tr>
<th>Time $t$</th>
<th>$n_0 = 1$</th>
<th>$n_0 = 100$</th>
<th>$n_0 = 1,000$</th>
<th>$n_0 = 10,000$</th>
<th>...</th>
<th>$n_0 \rightarrow \infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>31.985%</td>
<td>4.626%</td>
<td>3.506%</td>
<td>3.374%</td>
<td>...</td>
<td>3.359%</td>
</tr>
<tr>
<td>5</td>
<td>38.579%</td>
<td>5.790%</td>
<td>4.306%</td>
<td>4.336%</td>
<td>...</td>
<td>4.339%</td>
</tr>
<tr>
<td>10</td>
<td>47.188%</td>
<td>7.351%</td>
<td>5.856%</td>
<td>5.685%</td>
<td>...</td>
<td>5.665%</td>
</tr>
<tr>
<td>15</td>
<td>59.241%</td>
<td>9.477%</td>
<td>7.663%</td>
<td>7.457%</td>
<td>...</td>
<td>7.434%</td>
</tr>
<tr>
<td>20</td>
<td>78.060%</td>
<td>12.432%</td>
<td>10.029%</td>
<td>9.736%</td>
<td>...</td>
<td>9.725%</td>
</tr>
<tr>
<td>25</td>
<td>112.448%</td>
<td>16.811%</td>
<td>13.048%</td>
<td>12.610%</td>
<td>...</td>
<td>12.560%</td>
</tr>
<tr>
<td>30</td>
<td>190.275%</td>
<td>24.726%</td>
<td>16.965%</td>
<td>15.983%</td>
<td>...</td>
<td>15.870%</td>
</tr>
<tr>
<td>35</td>
<td>425.348%</td>
<td>46.751%</td>
<td>23.680%</td>
<td>19.957%</td>
<td>...</td>
<td>19.499%</td>
</tr>
</tbody>
</table>
Managing the longevity risk

Several tools can be developed to manage longevity risk. These tools can be placed and analysed in a risk management (RM) framework. The RM process consists of three basic steps, namely the identification of risks, the assessment (or measurement) of the relevant consequences, and the choice of the RM techniques. In what follows we refer to the RM process applied to life insurance, in general, and to life annuity portfolios, in particular.
Mortality/longevity risks belong to underwriting risks. Obviously, for an insurer the importance of the longevity risk (and so its existence) within the class of mortality risks is strictly related to the relative weight of the life annuity portfolio with respect to the overall life business.

Deterministic models can provide useful, although rough, insights into the impact of longevity risk on portfolio results. In particular, deterministic models allow us to calculate the range of values that some quantities (present value of future payments, annual outflows, or others) may assume in respect of the outcome of the underlying random quantity.

A rigorous assessment of the longevity risk requires the use of stochastic models. Risk management techniques for dealing with longevity risk include a wide set of tools, which can be interpreted, under an insurance perspective, as portfolio strategies, aimed at risk mitigation. A number of portfolio results can be taken as ‘metrics’ to assess the effectiveness of portfolio strategies (e.g. the annual outflows relating to annuity payments only, which constitute the starting point from which other quantities (e.g. profits) may be derived).

In the following figure we present a sequence of outflows, together with a barrier (the ‘threshold’) which represents a maintainable level of benefit payment.
The situation of annual outflows being above the threshold level, should be clearly avoided. To lower the probability of such critical situations, the insurer can resort to various portfolio strategies, in the framework of the RM process.

The following figure illustrates a wide range of portfolio strategies which aim at risk mitigation, in terms of lowering the probability and the severity of “longevity events”.

In practical terms, a Portfolio strategy can have as targets
(i) an increase in the maintainable annual outflow, and thus a higher threshold level;
(ii) lower (and smoother) annual outflows in the case of unanticipated improvements in portfolio mortality.

Both loss control and loss financing techniques (according to the RM language) can be adopted to achieve targets (i) and (ii).

Loss control techniques are mainly performed via the product design, that is, via an appropriate choice of the various items which constitute an insurance product. In particular, loss prevention is usually interpreted as the RM technique which aims to mitigate the loss frequency, whereas loss reduction aims at lowering the severity of the possible losses.

The pricing of insurance products provides a tool for loss prevention. This portfolio strategy is represented by path $(1) \rightarrow (a)$ in the figure. Referring to a life annuity product, the following issues, in particular, should be taken into account.

- Mortality improvements require the use of a projected life table for pricing life annuities.
- Because of the uncertainty in the future mortality trend, a premium formula other than the traditional one based on the equivalence principle should be adopted. It should be noted that, by adopting the equivalence principle, the longevity risk can be accounted for only via a (rough) safety loading, which is calculated by increasing the survival probabilities resulting from the projected life table. Indeed, this approach is often adopted in current actuarial practice.
- The presence, in an accumulation product such as an endowment, of an option to annuitize at a fixed annuitization rate requires an accurate pricing model accounting for the value of the option itself.
To pursue loss reduction, it is necessary to control the annuity amounts paid out. Hence, some flexibility must be added to the life annuity product.

One action could be the reduction of the annual amount as a consequence of an unanticipated mortality improvement (path (5) → (b) in the figure).

However, in this case the product would be a non-guaranteed life annuity, although possibly with a reasonable minimum amount guaranteed. A more practicable tool, consistent with the features of a guaranteed life annuity, consists of reducing the level of investment profit participation when the mortality experience is adverse to the annuity provider (path (4) → (b)). It is worth stressing that undistributed profits also increase the shareholders’ capital within the portfolio, hence increasing the maintainable threshold (path (3) → (a)).

Loss financing techniques require specific strategies involving the whole portfolio, and in some cases even other portfolios of the insurer. Risk transfer can be realized via (traditional) reinsurance arrangements (path (6) → (b)), swap-like reinsurance ((7) → (b)) and securitization, that is, Alternative Risk Transfer (ART). In the case of life annuities, ART requires the use of specific financial instruments, for example, longevity bonds ((8) → (b)), whose performance is linked to some measure of longevity in a given population. While mortality bonds (hedging the risk of a mortality higher than expected) already exist, longevity bonds (hedging the risk of a mortality lower than expected) are yet to appear in the market.

To the extent that mortality/longevity risks are retained by an insurer, the impact of a poor experience falls on the insurer itself. To meet an unexpected amount of obligations, an appropriate level of advance funding may provide a substantial help. To this purpose, shareholders’ capital must be allocated to the life annuity portfolio (path (2) → (a), as well as (3) → (a)), and the relevant amount should be determined to achieve insurer solvency.
Hedging strategies in general consist of assuming the existence of a risk which offsets another risk borne by the insurer. In some cases, hedging strategies involve various portfolios or lines of business (LOBs), or even the whole insurance company. In particular, natural hedging consists of offsetting risks in different LOBs. For example, writing both life insurance providing death benefits and life annuities for similar groups of policyholders may help to provide a hedge against longevity risk.

Clearly, mortality/longevity risks should be managed by the insurer through an appropriate mix of the tools described above. The choice of the RM tools is also driven by various interrelationships among the tools themselves.

**Natural hedging**

In the context of life insurance, natural hedging refers to a diversification strategy combining ‘opposite’ benefits with respect to the duration of life. The main idea is that if mortality rates decrease then life annuity costs increase while death benefit costs decrease (and vice versa). Hence the mortality risk inherent in a life annuity business could be offset, at least partially, by taking a position also on some insurance products providing benefits in the case of death. We discuss two situations, one concerning hedging across time and one across LOBs.

We first consider **hedging across time**. We assume that at time 0 (i.e. calendar year $x_0 + t_0$) an immediate life annuity is issued to a person aged $x_0$, with the proviso that at death (e.g. at the end of the year of death) the mathematical reserve therein set up to meet the life annuity benefit (only) is paid back to the beneficiaries. Reasonably, the reserving basis concerning the death benefit should be stated at policy issue so that the death benefit, although decreasing over time, is guaranteed.
At time 0, the random present value of future (life annuity and death) benefits for an individual (generically, individual $j$) is defined as follows:

$$Y_0^{(j)} = b^{(j)} \alpha_{x_0}^{(j)} + (1 + i)^{- (K_{x_0}^{(j)} + 1)} C_{K_{x_0}^{(j)} + 1}^{(j)}$$

With $C_{t}^{(j)} = b^{(j)} \alpha_{x_0 + t}^{(j)} = b^{(j)} \sum_{h=1}^{\omega - x_0 - t} (1 + i)^{-h} \psi_{x_0 + t}^{[A]}$. The benefit $C_{t}^{(j)}$ is therefore the mathematical reserve set up at time $t$ to meet the life annuity benefit, calculated according to the mortality assumption $A(\tau)$ and the annual interest rate $i$.

The individual reserve (meeting both the life annuity and the death benefit) to be set up at time $t$ according to the (traditional) equivalence principle is:

$$V_{t}^{(j)} = b^{(j)} \alpha_{x_0 + t}^{(j)} + \sum_{h=0}^{\omega - x_0 - t} \frac{h}{1} q_{x_0 + t} \left(1 + i\right)^{-(h+1)} C_{t + h + 1}^{(j)}$$

The sum at risk, $C_{t}^{(j)} - V_{t}^{(j)}$, in each year $(t - 1, t)$ is intended to be close to 0.

Intuitively, when dealing with both a life annuity and a death benefit the insurer benefits from a risk reduction, given that the longer is the annuity payment period, the lower is the amount of the death benefit. However, the risk reduction cannot be total, because of the definition of the death benefit. The tricky point of this package is the cost to the annuitant. The death benefit is expensive so commercial difficulties may arise.

Further solutions can be studied, in order to reconcile the risk reduction purposes of the insurer with the request by the annuitant for a high level of the ratio between the annual amount and the single premium. However, the lower is the death benefit, the lower is the risk reduction gained by the insurer. To give an example we consider a death benefit defined as the difference (if positive) between the single premium $S$ funding the life annuity benefit and the number of annual amounts paid up to death:

$$C_{t}^{(j)} = \max[S - (t - 1)b, 0]$$
One further, very well-known, example of natural hedging across time is given by reversionary annuities. In this case, the longer is the payment period to the leading annuitant, the lower should be the number of payments to the reversionary annuitant. However, some increased longevity risk arises in this case, due to the fact that two (or more) lives are involved instead of just one (with a possibly correlated mortality trend).

We now address natural hedging across LOBs. A risk reduction could be pursued by properly mixing positions in life insurances and life annuities. If the mortality rates of the two reference populations were perfectly correlated, there would be a perfect reduction of the risk but the offset result is unlikely to be as good as those mentioned previously, given that life insurances usually concern a different range of ages than life annuities (age basis risk). Further, we would point out that mortality trends emerge differently within life insurance and life annuity blocks of business (population basis risk).

The natural hedging across LOBs was first introduced by Cox and Lin (2007); Wang et al. (2009) introduce immunization strategies based on mortality duration and mortality convexity (similar to Redington immunization strategies for interest rate risk); Tsai et al. (2010) propose a natural hedging strategy obtained through the minimization of a risk measure (CVaR), while Li and Hardy (2011), use q-forward and the key q-duration instrument.

Zhu and Bauer (2015) investigate its real effectiveness in the light of models projections that capture changes in mortality that are distinct by age. The debate on natural hedging effectiveness is still open.
Natural hedging by Wang et al. (2009)

Wang et al. propose a longevity risk immunization strategy based on the analysis of mortality duration:

\[
D^V_\mu = \frac{dV}{d\mu} \cdot \frac{1}{V} \quad (\text{effective} \quad D^V_{e\mu} = \frac{V^+ - V^-}{2V\Delta\mu})
\]

and mortality convexity:

\[
C^V_\mu = \frac{d^2V}{d\mu^2} \cdot \frac{1}{V} \quad (\text{effective} \quad C^V_{e\mu} = \frac{V^+ + V^- - 2V}{V(\Delta\mu)^2})
\]

For a portfolio composed by annuities and term life insurance, it is possible to approximate the effect on the portfolio value of a change in mortality with:

\[
\Delta V = \left(\frac{dV^{life}}{d\mu} + \frac{dV^{ann}}{d\mu}\right)\Delta\mu + \left(\frac{d^2V^{life}}{d\mu^2} + \frac{d^2V^{ann}}{d\mu^2}\right)(\Delta\mu)^2
\]

The first order condition for natural hedging is:

\[
\left(\frac{dV^{life}}{d\mu} + \frac{dV^{ann}}{d\mu}\right)\Delta\mu = 0
\]

so

\[
D^{life}_\mu \frac{V^{life}}{V} + D^{ann}_\mu \frac{V^{ann}}{V} = 0
\]

Reinsurance arrangements

Various reinsurance arrangements can be conceived, at least in principle, to transfer longevity risk. Usually reinsurers are reluctant to accept such a transfer, due to the systematic nature of the risk of unanticipated aggregate mortality. Actually, only some slight offset (through natural hedging) can be gained by dealing with longevity risk just within the insurance-reinsurance process. Longevity-linked securities, transferring the risk to the capital market, could back the development of a longevity reinsurance market.
Stop-Loss reinsurance designed on annual outflows

A Stop-Loss reinsurance may be designed on annual outflows, the rationale is that, at a given point in time, longevity risk is perceived if the amount of benefits to be currently paid to annuitants is (significantly) higher than expected. A transfer arrangement can then be designed so that the reinsurer takes charge of such an extra amount, or ‘loss’. The loss may be due to random fluctuation. By setting a trigger level for the reinsurer’s intervention higher than the expected value of the amount of benefits, we would reduce the possible transfer of such a random risk component.

Reinsurance conditions should concern the following items:

- Let z be the time of issue (or revision) of the arrangement. The time horizon k of the reinsurance coverage should be stated, as well as the timing of the possible reinsurer’s intervention within it. Within the time horizon k, policy conditions (i.e. premium basis, mortality assumptions, and so on) should be guaranteed. As to the timing of the intervention of the reinsurer, since reference is to annual outflows, it is reasonable to assume that a yearly timing is chosen.
- The mortality assumption for calculating the expected value of the outflow, required to define the loss of the cedant. Reasonably, we will adopt the current mortality table, which will be generically denoted as \( A(\tau) \) in what follows.
- The minimum amount \( \Lambda'_t \) of benefits (at time \( t, t = z + 1, z + 2, ..., z + k \)) below which there is no payment by the reinsurer. For example,

\[
\Lambda'_t = \mathbb{E}[R_t](1 + r) = \mathbb{E}[B^T_t|A(\tau), n_z](1 + r) = b\mathbb{E}[N_t|A(\tau), n_z](1 + r)
\]

- The Stop-Loss upper limit, that is, an amount \( \Lambda''_t \) such that \( \Lambda''_t - \Lambda'_t \) is the maximum amount paid by the reinsurer at time \( t \). From the point of view of the cedant, the amount \( \Lambda''_t \) should be set high enough so that only situations of extremely high survivorship are charged to the cedant. However, the reinsurer
reasonably sets $\Lambda''_t$ in connection to the available hedging opportunities. As to the cedant, a further reinsurance arrangement may be underwritten, if available, for the residual risk, possibly with another reinsurer; in this case, the amount $\Lambda''_t - \Lambda'_t$ operates as the first layer.

In the following figure a typical situation is represented.

We now define in detail the flows paid by the reinsurer. Let $B_{t_t}^{(SL)}$ denote such flow at time $t$, $t = z + 1, z + 2, \ldots, z + k$. We have

$$B_{t_t}^{(SL)} = \begin{cases} 0 & \text{if } B_{t_t}^{(SL)} \leq \Lambda'_t \\ B_{t_t}^{(SL)} - \Lambda'_t & \text{if } \Lambda'_t < B_{t_t}^{(SL)} \leq \Lambda''_t \\ \Lambda''_t - \Lambda'_t & \text{if } B_{t_t}^{(SL)} > \Lambda''_t \end{cases}$$

The net outflow of the cedant at time $t$ (gross of the reinsurance premium), denoted as $OF_{t_t}^{(SL)}$ is then

$$OF_{t_t}^{(SL)} = B_{t_t}^{(SL)} - B_{t_t}^{(SL)} = \begin{cases} 0 & \text{if } B_{t_t}^{(SL)} \leq \Lambda'_t \\ \Lambda'_t & \text{if } \Lambda'_t < B_{t_t}^{(SL)} \leq \Lambda''_t \\ B_{t_t}^{(SL)} - (\Lambda''_t - \Lambda'_t) & \text{if } B_{t_t}^{(SL)} > \Lambda''_t \end{cases}$$
The net outflow of the cedant is clearly random but, unless some ‘extreme’ survivorship event occurs, it is protected with a cap. It is interesting (especially for comparison with the swap-like arrangement described subsequently) to comment on this outflow. First of all, it must be stressed that $B_t^\Pi \leq \Lambda'_t$ represents a situation of profit or small loss to the insurer. On the contrary, the event $B_t^\Pi > \Lambda''_t$ corresponds to a huge loss. Whenever $\Lambda'_t < B_t^\Pi \leq \Lambda''_t$ a loss results for the insurer, whose severity may range from small (if $B_t^\Pi$ is close to $\Lambda'_t$ to high (if $B_t^\Pi$ is close to $\Lambda''_t$). So the effect of the Stop-Loss arrangement is to transfer to the reinsurer all of the loss situations, except for the lowest and the heaviest ones; any situation of profit, on the contrary, is kept by the cedant.

**Reinsurance-swap arrangement on annual outflows**

To reduce further randomness of the annual outflow, the cedant may be willing to transfer to the reinsurer not only losses, but also profits. Thus, a reinsurance-swap arrangement on annual outflows can be designed. Let $B_t^*$ be a target value for the outflows of the insurer at time $t$, $t = z + 1, z + 2, \ldots, z + k$; for example,

$$B_t^* = \mathbb{E}[B_t^\Pi | A(\tau), n_z]$$

where $A(\tau)$ is an appropriate mortality assumption and $z$ is the time of issue of the reinsurance swap. Under the swap, if $B_t^\Pi > B_t^*$ the cedant receives money from the reinsurer; otherwise, if $B_t^\Pi < B_t^*$, then the cedant gives money to the reinsurer, so that the target outflow is reached.

Let $B_t^{(swap)}$ be the payment from the reinsurer to the cedant, defined as follows:

$$B_t^{(swap)} = B_t^\Pi - B_t^*$$

The annual outflow (gross of the reinsurance premium) for the cedant at time $t$ is

$$OF_t^{(swap)} = B_t^\Pi - B_t^{(swap)} = B_t^*$$
The advantage for the cedant is to convert a random flow, $B_t^{\Pi}$, into a certain flow, $B_t^*$ and hence the term ‘reinsurance-swap’ that we have assigned to this arrangement. The following figure depicts a possible situation.

![Graph](image)

Note that, ceteris paribus, this arrangement should be less expensive than the Stop-Loss treaty on outflows, given that the reinsurer participates not only in the losses, but also in the profits.

The design of the reinsurance-swap can be generalized by assigning two barriers $\Lambda'_t$, $\Lambda''_t$ (with $\Lambda'_t \leq B_t^* \leq \Lambda''_t$) obtaining a “collar” such that

$$B_t^{(\text{swap-b})} = \begin{cases} 
B_t^{(\Pi)} - \Lambda'_t & \text{if } B_t^{(\Pi)} \leq \Lambda'_t \\
0 & \text{if } \Lambda'_t < B_t^{(\Pi)} \leq \Lambda''_t \\
B_t^{(\Pi)} - \Lambda''_t & \text{if } B_t^{(\Pi)} > \Lambda''_t
\end{cases}$$

Clearly, when setting $\Lambda'_t = B_t^* = \Lambda''_t$ one finds the reinsurance swap arrangement again. The net outflow (gross of the reinsurance premium) to the cedant is then
It is interesting this last expression with the Stop-Loss arrangement. We have already commented on the implications of Stop-Loss for the profit/loss left to the cedant. Under the “collar” arrangement, large losses as well as large profits are transferred to the reinsurer; therefore, both a floor and a cap are now applied to the profits/losses of the cedant.

**Longevity risk securitization**

Through the securitization the Longevity risk is transferred to the capital markets.

Securitization is a process that consists in isolating the sources of risk and the financial flows generated by them and "packaging" them in financial instruments (longevity / mortality-linked securities) traded on the capital markets.

His main advantages compared to reinsurance are: a potential greater capacity for underwriting by the markets, lower costs.

The longevity/mortality-linked securities proposals in the literature are:

- Longevity and mortality bonds
- Survivor and mortality swaps
- Mortality futures
- Mortality options

Blake and Burrows (2001) were the first to support the use of mortality/(longevity) - linked securities for the transfer of longevity risk to the capital market.
The difficulty in affirming the new market are:

- scale imbalance between existing exposures and the willingness of hedge providers;
- mortality-linked securities must meet the different needs of hedgers and investors (difficult to reconcile - the former require the effectiveness of hedging, while the latter demand liquidity);
- absence of a market price for the risk of longevity.

Loeys et al. (2007) explain that in order to establish a new successful capital market, it must:

- provide effective exposure or coverage (the lower the number of contracts traded, the greater the liquidity in each contract, but the lower the effectiveness of the hedge, national indices and base risk);
- at a risk of the world that is economically important (good balance between longevity supply and demand: this will affect the overall size of the market, as well as the price of longevity risk - The involvement of capital markets will reduce longevity risk management costs: capacity increase, together with greater price transparency (arbitrage players) and greater liquidity (speculators));
- that cannot be covered through the tools of the existing market;
- use a homogeneous and transparent contract to allow exchanges between agents.

**Mortality bonds**

Are short-term bond, they represent market-marketable securities whose payments are linked to a mortality index. They are similar to catastrophe bonds. Ex. Swiss Re mortality bond issued in December 2003. It belongs to the principal at risk type. In particular, the maturing capital is reduced by 5% for every 1% of growth of a specified mortality index linked to different populations (USA, UK, France, Italy, Switzerland),
exceeding the threshold of 1.3 at maturity. The assumption of risk by investors is repaid by means of coupons that recognize a spread with respect to the US-LIBOR.

**Longevity bonds (or survivor bonds)**

They could be of two type: principal-at-risk longevity bonds or coupon based longevity bonds. They can have a fixed expiry or a stochastic expiration.

Es EIB - BNP Paribas 2004 (if mortality had been lower than expected, the coupons on the bond would have decreased and vice versa). It presented:

- design shortcomings (the basic risk was too big);
- issue related to pricing of longevity risk;
- institutional shortcomings (the size of the issue was too small to create a liquid market).

Lessons: need for a good set of objectively calculated, transparent and relevant mortality rates (LifeMetrics); more transparent stochastic forecast models of mortality.

**Possible structure of a longevity bond**

Obligation of the pension provider:

- immediate annuity to a cohort of \(l_{x_0}\) annuitants aged \(x_0\) at time 0;
- \(R\) is the annual amount payed by the annuity;
- in \(t\) the pension provider will pay the random amount \(Rl_{x_0+t}\);
- so he is exposed to the risk of systematic deviation of \(l_{x_0+t}\) from is expected value \(\hat{l}_{x_0+t}\).

We consider a straight coupon bond that offers an aggregate cash flow \(RC_t\) at each time \(t\) and pays an amount \(RF\) at the maturity. Lin and Cox (2005) assume constant coupon \(RC\).
The following picture represents the cash flows of the hedging strategy.

Through the Special Purpose Company the flow of coupons $C$ is divided between the pension provider and investors:

$$ R \cdot C = R \cdot (B_t + D_t) $$

The payments at the annuity provider are assumed to be:

$$ B_t = \begin{cases} 
C & l_{x_0 + t} - \hat{l}_{x_0 + t} > C \\
 l_{x_0 + t} - \hat{l}_{x_0 + t} & 0 < l_{x_0 + t} - \hat{l}_{x_0 + t} \leq C \\
0 & l_{x_0 + t} - \hat{l}_{x_0 + t} \leq 0 
\end{cases} $$

While the payments to the investors are:

$$ D_t = C - B_t $$

$$ D_t = \begin{cases} 
0 & l_{x_0 + t} - \hat{l}_{x_0 + t} > C \\
 C - (l_{x_0 + t} - \hat{l}_{x_0 + t}) & 0 < l_{x_0 + t} - \hat{l}_{x_0 + t} \leq C \\
 C & l_{x_0 + t} - \hat{l}_{x_0 + t} \leq 0 
\end{cases} $$
Prices are obtained as expected present value of the future payoff under a risk adjusted probability measure.

The price \( W \) paid by the SPC to buy the straight bond is:

\[
W = RFd(0,T) + RC \sum_{t=1}^{T} d(0,t)
\]

The premium \( P \) paid by the pension provider in order to obtain protection from longevity risk is:

\[
P = R \sum_{t=1}^{T} \tilde{E}[B_t]d(0,t)
\]

The price \( V \) paid by the investors to buy the longevity bond is:

\[
V = RFd(0,T) + R \sum_{t=1}^{T} \tilde{E}[D_t]d(0,t)
\]

with \( d(0,t) \) discount factor

and \( \tilde{E} \) expectation under the risk adjusted probability measure.

\textit{Note: we assume independence of demographic and financial risk.}

\textbf{Longevity derivatives}

Longevity derivatives are capital market instruments that have payoffs linked to the level of a mortality index. The subjects interested in creating a market for longevity derivatives are hedgers, institutional investors, speculators, arbitrageurs, government, regulators. The derivatives proposed in literature are:

- q-Forwards
- survivor swaps
• mortality options

q-forwards

It is the simplest longevity derivative. A q-forward is an agreement between two parties for the exchange at a future date of an amount proportional to the realized mortality rate of a given population against an amount proportional to a fixed mortality rate mutually agreed at the beginning. It is a zero-coupon swap that exchanges fixed mortality and mortality achieved on maturity.

The pension provider must take a short position (floating payer, q-forward seller), while the investor is the q-forward buyer.

Let \( V \) be the current value (in \( t = 0 \)) of the cash flows paid to policyholders, and \( q \) the stochastic vector of the probability of death of the interest cohort and \( E(q) \) its best estimate.

In the absence of hedging, the loss present value is given by \( L = V(q) - V(E(q)) \).

In case of hedging, the loss is given by \( L_h = L - hH \), where \( h \) is the number of q-forward held and \( H \) the present value of the payoff of the q-forward \( ((1 + r)^{-t}[q_f(x, t) - q(x, t)]) \)

As a hedging objective, let's minimize the variance of \( L - hH: \min\{\sigma^2(hH - L)\} \)

The optimal hedge ratio is given by: \( h = \frac{\text{cov}(L, H)}{\sigma^2(H)} = \rho_{L,H} \frac{\sigma(L)}{\sigma(H)} \).

It is possible to use more forward, theoretically as many as the deadlines of the cash flows.

Survivor swaps

Survivor swaps (basic) are agreement between two counterparties to exchange a fixed cash flow in the future in exchange of a single random cash flow dependent on mortality (Dowd et al. 2006). The cash flows are linked to the number of survivors of
a given cohort. Compared to other mortality-linked securities - e.g. longevity bonds - the survivor swaps have some advantages:

- They result in lower transaction costs
- They are more flexible and tailor-made to meet the various needs
- They do not require the existence of a liquid market

Vanilla survivor swaps (set of basic survivor swaps) are counterparties agreement to exchange a series of periodic payments (for each \( t = 1,2,\ldots,S \) until the expiration of the swap \( S \)). Fixed leg depends on the expected survivors of a given cohort while the floating leg depends on actual survivors on future maturities.

The structure of a survivor swap is represented in the following:

- the annuity provider must pay immediate annuities to a cohort of \( l_x \) annuitants aged \( x \) at time zero;
  - we assume a fixed annuity of € 1;
  - \( \hat{l}_{x+t} \) is the expected number of survivors aged \( x + t \) at time \( t \);
  - \( l_{x+t} \) is the actual number of survivors aged \( x + t \) at time \( t \);
- The annuity provider is exposed to the systematic risk of deviations between \( l_{x+t} \) and \( \hat{l}_{x+t} \);
- \( l_{x+t} - \hat{l}_{x+t} \) are the losses suffered by the annuity provider at each deadline \( t \);
- \( \pi \): is the swap rate fixed so that the swap value is zero at issue: fixed leg market value = floating leg market value

![Diagram of Survivor Swap](image)
The value of the vanilla survivor swap at time zero for the fixed-rate payer is:

\[ V[l_{x+t}] - V[(1 + \pi)\hat{l}_{x+t}] \]

assuming independence of demographic and financial risk

\[ V[(1 + \pi)\hat{l}_{x+t}] = (1 + \pi)\sum_{t=1}^{T} \hat{l}_{x+t} d(0,t) \]

is the expected present value of the fixed leg under the probability measure of the real world, and

\[ V[l_{x+t}] = \sum_{t=1}^{T} E^*(l_{x+t}) d(0,t) \]

is the expected present value of the floating leg under the risk-adjusted probability measure.

The \( \pi \) that guarantee \( V[l_{x+t}] - V[(1 + \pi)\hat{l}_{x+t}] = 0 \) is:

\[ \pi = \frac{\sum_{t=1}^{T} E^*(l_{x+t}) d(0,t)}{\sum_{t=1}^{T} \hat{l}_{x+t} d(0,t)} - 1 \]

In the following figure a possible situation is represented: